SUPER-BROWNIAN MOTION
WITH REFLECTING HISTORICAL PATHS

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Abstract. We consider super-Brownian motion whose historical paths reflect from each other, unlike those of the usual historical super-Brownian motion. We prove tightness for the family of distributions corresponding to a sequence of discrete approximations but we leave the problem of uniqueness of the limit open. We prove a few results about path behavior for processes under any limit distribution. In particular, we show that for any $\gamma > 0$, a "typical" increment of a reflecting historical path over a small time interval Δt is not greater than $(\Delta t)^{3/4-\gamma}$.

1. Introduction.

The present article has been inspired by two probabilistic models—superprocesses with interactions and reflected particle systems.

The first person to study a reflecting system of particles was Harris [H] who considered an infinite system of Brownian particles on the line. He proved that if the initial positions of the particles are points of a Poisson point process, then for a large time t the distribution of a single particle is normal with the standard deviation $(2t/\pi)^{1/4}$. Spitzer [S] analyzed a similar model with particles moving along straight lines between collisions. See [DGL1, DGL2, G, Ho] for related results.

The simplest superprocesses, for example, super-Brownian motion, are continuum limits of branching systems in which the branching mechanism is independent of the positions of particles. There has been considerable activity studying models with interactions. Many articles are devoted to models with catalysts, see, e.g., [DF, De]. Various other models with interactions are discussed in [AT, BHM, EP, P3]. See in particular [P4] and references therein.

We will study a model similar to that introduced by Harris, in that we will start with linear Brownian motion as the spatial process. We will attempt to build a corresponding

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superprocess with historical paths that do not cross over although they may touch each other.

Our construction is based on a sequence of discrete approximations. Consider for every $\varepsilon \in (0,1]$ a branching particle system which starts initially with N_ε particles located respectively at $x_1^\varepsilon \leq \cdots \leq x_{N_\varepsilon}^\varepsilon$. Particles move independently in space according to linear Brownian motion and are subject to critical binary branching at rate ε^{-1}. To be specific, the lifetimes of the particles are exponential with parameter ε^{-1} and when a particle dies it gives rise to 0 or 2 new particles with probability 1/2.

Let us now introduce our basic assumptions. Let

$$\mu_\varepsilon := \varepsilon \sum_{j=1}^{N_\varepsilon} \delta_{x_j^\varepsilon}$$

and assume that there is a finite measure μ on $\mathbb{R}$ such that

$$\mu_\varepsilon \xrightarrow[\varepsilon\to 0]{\text{(w)}} \mu, \tag{1.1}$$

where the notation (w) indicates weak convergence in the space $M_f(\mathbb{R})$ of finite measures on $\mathbb{R}$. In addition, if $\operatorname{supp}\mu$ denotes the topological support of μ, we assume that

$$\operatorname{supp}\mu_\varepsilon \xrightarrow[\varepsilon\to 0]{} \operatorname{supp}\mu, \tag{1.2}$$

in the sense of the Hausdorff metric on compact subsets of $\mathbb{R}$ (in particular, we assume that $\operatorname{supp}\mu$ is compact).

Let X_t^ε denote the random measure equal to ε times the sum of the Dirac point masses at the positions of particles alive at time t. Then,

$$(X_t^\varepsilon, t \geq 0) \xrightarrow[\varepsilon\to 0]{\text{(d)}} (X_t, t \geq 0), \tag{1.3}$$

where the limit process is super-Brownian motion in $\mathbb{R}$ with branching rate $\gamma = 1$ (throughout this work we consider only this branching rate) and initial value μ, and the convergence holds in distribution in the Skorohod space $\mathbf{D}(\mathbb{R}_+, M_f(\mathbb{R}))$. The convergence (1.3) is the standard approximation of super-Brownian motion (see e.g. [P4]). Note that assumption (1.2) is not needed for (1.3) but it guarantees that the graph of X^ε also converges in distribution to the graph of X (see Lemma 2.3 below), a property that plays an important role in our arguments.

For each particle alive at time t, we can consider its historical path, which is the element of $\mathbf{C}([0,t], \mathbb{R})$ obtained by concatenating the trajectories of the ancestors of the given particle up to time t. Denote by Y_t^ε the historical measure equal to ε times the sum

of the Dirac point masses at the historical paths of the particles alive at time t (Y_t^ε is thus a random measure on the set $\mathbf{C}([0,t],\mathbb{R})$ of continuous mappings from $[0,t]$ into $\mathbb{R}$). Then the convergence (1.3) can be reinforced as

$$(Y_t^\varepsilon, t \geq 0) \xrightarrow[\varepsilon\to 0]{\text{(d)}} (Y_t, t \geq 0), \tag{1.4}$$

where the limit process is now historical super-Brownian motion started at μ.

For every $\varepsilon > 0$, we can use the original branching particle system to construct a new system with reflection. The branching mechanism (critical binary branching at rate ε^{-1}) is the same as in the original system, but the particle paths in the new system reflect against each other. A precise construction is given in Section 3, but let us give an informal description. The reflected system is such that for every $t \geq 0$, the set of positions of particles at time t is the same as in the original system, and in particular the branching times are the same. During the time interval between 0 and the first branching time, the vector of positions of the particles labeled $1, 2, \dots, N_\varepsilon$ in the reflected system is the increasing rearrangement of the vector of positions of the particles in the original system. Suppose that at the first branching time, denoted by ξ, a particle dies and gives rise to 2 children. If the location of this particle is the j-th coordinate in the increasing rearrangement of the vector of positions at time $\xi-$, we will say that in the reflected system particle j has given rise to two children labeled $j1$ and $j2$. Then on the interval between ξ and the second branching time, the vector of positions of the particles labeled $1, \dots, j-1, j1, j2, j+1, \dots, N_\varepsilon$ in the reflected system is again the increasing rearrangement of the vector of positions of the particles in the original system. We can easily continue this construction by induction.

Denote by $\widetilde{X}_t^\varepsilon$ and $\widetilde{Y}_t^\varepsilon$ the analogues of X_t^ε and Y_t^ε for the the system with reflection. We have $\widetilde{X}_t^\varepsilon = X_t^\varepsilon$ since the set of positions of particles is the same at every time t in both systems. On the other hand, $\widetilde{Y}_t^\varepsilon$ is typically very different from Y_t^ε. Indeed, the following property holds for any two paths w, w' in the support of $\widetilde{Y}_t^\varepsilon$: Either $w(r) \leq w'(r)$ for every $0 \leq r \leq t$, or $w(r) \geq w'(r)$ for every $0 \leq r \leq t$.

The main purpose of this work is to try to understand the limiting behavior of the branching particle system with reflection as $\varepsilon \to 0$. Our primary objective was to get an analogue of the convergence (1.4) when the processes Y^ε are replaced by $\widetilde{Y}^\varepsilon$, giving information about the individual paths in the system with reflection. We did not completely succeed in this task, but we can prove the following result, where $\mathcal{W}$ denotes the set of all stopped paths, or equivalently the union over all $t \geq 0$ of the sets $\mathbf{C}([0,t],\mathbb{R})$.

Theorem 1.1. *Let $\mathcal{E}$ be a sequence of positive numbers converging to* 0*. The laws of the processes* $(\widetilde{Y}^\varepsilon_t, t \geq 0)$ *for* $\varepsilon \in \mathcal{E}$ *are tight in the space of all probability measures on the Skorohod space* $\mathbf{D}([0,\infty), M_f(\mathcal{W}))$*. Furthermore any limiting distribution is supported on* $\mathbf{C}([0,\infty), M_f(\mathcal{W}))$.

Hence, by extracting a subsequence if necessary, we can assume that the sequence of processes $\widetilde{Y}^\varepsilon$ converges in distribution towards a process $\widetilde{Y}$ with continuous paths with values in $M_f(\mathcal{W})$. Note that, for every $t \geq 0$, the measure $\widetilde{Y}_t$ is supported on $\mathbf{C}([0,t],\mathbb{R})$. Although the question of uniqueness of the limit remains unsolved, we are able to derive several results on the path behavior of the process $\widetilde{Y}$.

First note that, since $\widetilde{X}^\varepsilon_t = X^\varepsilon_t$ for every $t \geq 0$, the convergence (1.1) implies that the $M_f(\mathbb{R})$-valued process $\widetilde{X}$ defined by

$$\langle \widetilde{X}_t, \varphi \rangle = \int \widetilde{Y}_t(dw)\, \varphi(w(t))$$

is a super-Brownian motion started at μ. In particular, it is known (see [KS], [R]) that a.s. for every $t > 0$ the measure $\widetilde{X}_t(dy)$ has a density denoted by $x_t(y)$, and that there exists a jointly continuous modification of $(x_t(y), t > 0, y \in \mathbb{R})$.

The next result shows that for any $\gamma > 0$, a typical oscillation of a reflecting historical path is not greater than $(\Delta t)^{\frac{3}{4}-\gamma}$, and hence much smaller than a typical Brownian oscillation $(\Delta t)^{\frac{1}{2}}$. This result is consistent with the Harris [H] estimate, if we translate the large-time asymptotics to small-time asymptotics.

Theorem 1.2. *Almost surely for every* $t > 0$ *and every* $r \in (0,t)$*, for every path* $w \in \operatorname{supp} \widetilde{Y}_t$*, the condition* $x_r(w(r)) > 0$ *implies that, for every* $\gamma > 0$*,*

$$\limsup_{\delta \downarrow 0} \frac{|w(r+\delta) - w(r)|}{\delta^{\frac{3}{4}-\gamma}} = 0.$$

A more precise version of Theorem 1.2 is given in Section 5 (Theorem 5.10). It is not hard to check that if we fix $t > 0$ and $r \in (0,t)$ (fixing r is in fact enough), the condition $x_r(w(r)) > 0$, and thus the conclusion of the theorem, will hold for every path $w \in \operatorname{supp} \widetilde{Y}_t$, a.s. Alternatively, for every fixed $t > 0$, the conclusion of Theorem 1.2 holds for a set of values of $r \in (0,t)$ of full Lebesgue measure, for every $w \in \operatorname{supp} \widetilde{Y}_t$. We believe that $\delta^{\frac{3}{4}}$ is the “typical” size for the oscillation of a historical reflected path although we have no lower bound justifying this claim.

We also study the behavior of reflected historical paths at a branching point. If w and w' are two reflected historical paths that coincide up to time $r > 0$ (meaning informally that the corresponding “particles” have the same ancestor up to time r), we show that the

distance between $w(r+\delta)$ and $w'(r+\delta)$ grows linearly as a function of δ, up to logarithmic corrections. The precise statement is as follows.

Theorem 1.3. *Let $t > 0$. If w and w' are two distinct elements of $\mathbf{C}([0,t],\mathbb{R})$, we set*

$$\gamma_{w,w'} = \inf\{r \geq 0 : w(r) \neq w'(r)\}.$$

Then a.s. for any two distinct paths $w, w' \in \operatorname{supp} \widetilde{Y}_t$ such that $\gamma_{w,w'} > 0$, we have

$$\limsup_{\delta \downarrow 0} \frac{|w(\gamma_{w,w'} + \delta) - w'(\gamma_{w,w'} + \delta)|}{2\delta \log|\log \delta|} = x_{\gamma_{w,w'}}(w(\gamma_{w,w'})) > 0$$

and, for every $\gamma > 0$,

$$\lim_{\delta \downarrow 0} \frac{|w(\gamma_{w,w'} + \delta) - w'(\gamma_{w,w'} + \delta)|}{\delta|\log \delta|^{-1-\gamma}} = \infty.$$

Our proofs rely on several known results on super-Brownian motion. In particular, we use the Brownian snake idea [L2] in an essential way, both in the proofs and for giving more precise versions of the results. For instance, as a key step towards Theorem 1.1, we get a uniform continuity result (Theorem 4.1) for the historical paths of the approximating branching particle systems with reflection. The proof of this result requires some precise information about the genealogical structure of the approximating systems, which seems to be more easily accessible via the snake approach (cf Lemma 2.1 below).

For an introduction to the theory of superprocesses (measure-valued diffusions) and historical processes, the reader may consult [Da, Dy, DP, L2, P4].

The paper is organized as follows. Section 2 describes the specific coding that we use to represent the genealogical structure of the approximating branching particle systems. This section also contains a few important preliminary results. Section 3 presents the construction of the systems with reflection. Tightness results are given in Section 4, including a more precise form of Theorem 1.1. Section 5 contains the proof of Theorem 1.2, and is the most technical part of the paper. Finally, Theorem 1.3 is proved in Section 6.

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2. Coding discrete trees.

We will describe a method that provides a coding of the genealogy of the branching particle systems introduced in Section 1, in a consistent way for all values of the parameter $\varepsilon \in (0, 1]$. This method involves embedding branching trees in a path of reflected Brownian motion, and is based on [L1] (see also [NP]).

2.1 Markov chains embedded in reflected Brownian motion.

Let $\beta = (\beta_s, s \geq 0)$ be distributed as twice a reflected Brownian motion on $\mathbb{R}_+$:

$$(\beta_s, s \geq 0) \overset{\text{(d)}}{=} (2|B_s|, s \geq 0),$$

where B is a standard linear Brownian motion, with $B_0 = 0$. The reason for the factor 2 will be clear later. We denote by $(L_s^x, x \geq 0, s \geq 0)$ the jointly continuous family of local times of β, normalized in such a way that, for every nonnegative Borel function φ on $\mathbb{R}_+$,

$$\int_0^t \varphi(\beta_s)\, ds = \int_{\mathbb{R}_+} \varphi(x) L_t^x dx.$$

Also set $\tau_r = \inf\{s \geq 0 : L_s^0 > r\}$, for every $r > 0$.

For every $\varepsilon \in (0,1]$, we introduce a sequence of stopping times $(T_k^\varepsilon, k = 0, 1, \ldots)$ defined inductively as follows:

$$\begin{aligned} T_0^\varepsilon &= \inf\{s \geq 0 : \beta_s = 2\varepsilon\}, \\ T_{2k+1}^\varepsilon &= \inf\{u \geq T_{2k}^\varepsilon : \sup_{T_{2k}^\varepsilon \leq s \leq u} \beta_s - \beta_u = 2\varepsilon\}, \\ T_{2k+2}^\varepsilon &= \inf\{u \geq T_{2k+1}^\varepsilon : \beta_u - \inf_{T_{2k+1}^\varepsilon \leq s \leq u} \beta_s = 2\varepsilon\}. \end{aligned}$$

It is simple to check that the variables $T_0^\varepsilon, T_1^\varepsilon - T_0^\varepsilon, T_2^\varepsilon - T_1^\varepsilon, \ldots$ are independent and identically distributed. To see this, note that if $(\gamma_t, t \geq 0)$ is a reflected Brownian motion with initial value $\gamma_0 = b \geq 0$, the process

$$\gamma_t - \inf_{0 \leq s \leq t} \gamma_s$$

is again a reflected Brownian motion, with initial value 0, and also observe that $\beta_{T_{2k}^\varepsilon} \geq 2\varepsilon$ for every k.

As $E(T_0^\varepsilon) = \varepsilon^2$, standard arguments show that for every $K > 0$

$$\sup_{s \leq K} \left| T_{[s/\varepsilon^2]}^\varepsilon - s \right| \xrightarrow[\varepsilon \to 0]{\text{a.s.}} 0. \tag{2.1}$$

(First establish this convergence along the sequence $\varepsilon_n = n^{-2}$ and then use monotonicity arguments.) Thus,

$$\sup_{s \leq K} \left| \beta_{T_{[s/\varepsilon^2]}^\varepsilon} - \beta_s \right| \xrightarrow[\varepsilon \to 0]{\text{a.s.}} 0. \tag{2.2}$$

For $k = 0, 1, \ldots$, set

$$\begin{aligned} S_{2k}^\varepsilon &= \beta_{T_{2k}^\varepsilon} - 2\varepsilon, \\ S_{2k+1}^\varepsilon &= \beta_{T_{2k+1}^\varepsilon}. \end{aligned}$$

It is easy to verify that $(S^\varepsilon_k, k = 0, 1, 2, \ldots)$ is a time-inhomogeneous Markov chain with values in $\mathbb{R}_+$, whose law can be described as follows (see [L1] Section 3 for details): $S^\varepsilon_0 = 0$ and S^ε_{2k+1} has the same distribution as $S^\varepsilon_{2k} + U$, where U is an exponential variable with mean 2ε, independent of S^ε_{2k}, S^ε_{2k+2} has the same distribution as $(S^\varepsilon_{2k+1} - V)_+$ where V is exponential with mean 2ε, independent of S^ε_{2k+1}.

From (2.2), we have a.s. for every $K > 0$,

$$\sup_{s\le K}\left|S^\varepsilon_{[s/\varepsilon^2]} - \beta_s\right| \xrightarrow[\varepsilon\to 0]{\text{a.s.}} 0.$$

We then define a continuous-time process $(\beta^\varepsilon_s, s \ge 0)$ by setting

$$\beta^\varepsilon_{k\varepsilon^2} = S^\varepsilon_k \text{ for } k = 0, 1, 2, \ldots$$

and by interpolating linearly on intervals of the form $[k\varepsilon^2, (k+1)\varepsilon^2]$. It is obvious that we also have

$$\sup_{s\le K}|\beta^\varepsilon_s - \beta_s| \xrightarrow[\varepsilon\to 0]{\text{a.s.}} 0. \tag{2.3}$$

2.2 The correspondence between excursions and trees

With each excursion of β^ε away from 0, we can associate a marked tree representing the genealogical structure of a Galton-Watson branching process with critical binary branching at rate ε^{-1}, starting with one individual (the ancestor) at time 0. Here a marked tree consists of the set $\mathcal{T}$ of edges (i.e., particles), which is a subset of

$$\mathbf{U} := \bigcup_{n=0}^{\infty}\{1, 2\}^n \qquad \text{(by convention, } \{1, 2\}^0 = \{\emptyset\}\text{)},$$

and the family $(\ell_u, u \in \mathcal{T})$ of lengths of edges (i.e., lifetimes of particles).

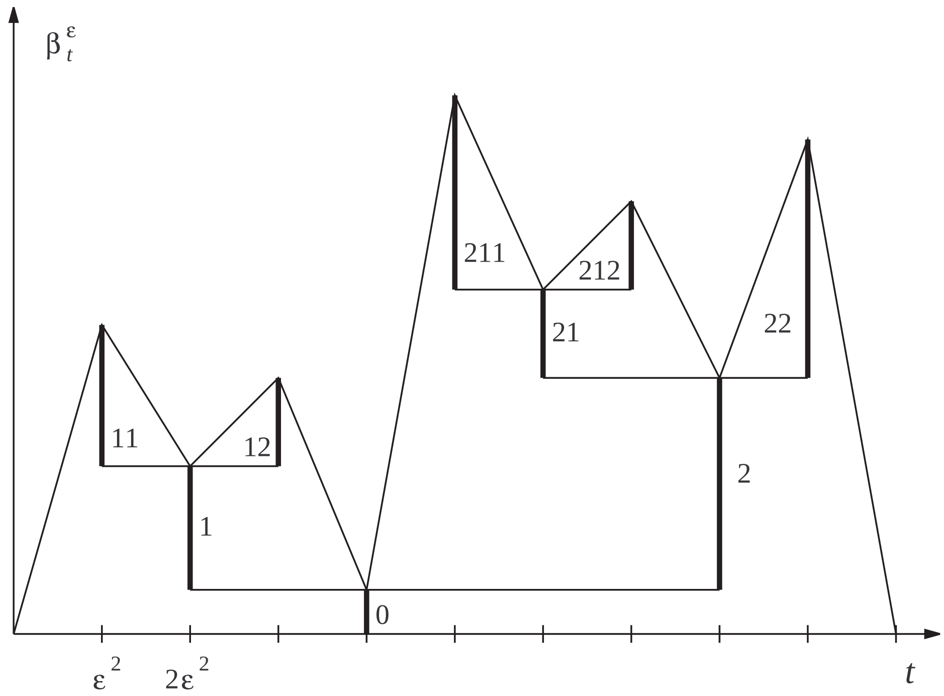

Figure 1.

This correspondence is explained in Fig. 1 for the first excursion of β^ε away from 0. Informally, if $(i\varepsilon^2, j\varepsilon^2)$ is the interval corresponding to an excursion of β^ε, the lifetime $\ell_\emptyset$ of the individual at the root of the associated tree is the minimum of β^ε over $[(i{+}1)\varepsilon^2, (j{-}1)\varepsilon^2]$ and this individual has two children if and only if $j{-}1 > i{+}1$. In that case, by decomposing the excursion restricted to $[(i{+}1)\varepsilon^2, (j{-}1)\varepsilon^2]$ at the time of its minimum over this interval, we get two new excursions, each of which codes the genealogical structure of descendants of one of the ancestor's children. The construction of the tree is then completed by induction. Note that each time of the form $k\varepsilon^{-2}$ in the interval $(i\varepsilon^2, j\varepsilon^2)$ corresponds to one edge of the tree (for instance the time of the minimum over $[(i+1)\varepsilon^2, (j-1)\varepsilon^2]$ corresponds to $\emptyset$, see Fig. 1). We refer to [L1] Section 2 for a more precise description and a proof that this construction yields the family tree of a Galton-Watson branching process with critical binary branching at rate ε^{-1}. (We can now explain the factor 2 in the definition of β: We want the branching rate to be ε^{-1} and not $(\varepsilon/2)^{-1}$.)

There is a one-to-one correspondence between excursions of β^ε away from 0 and excursions of β away from 0 with height greater than 2ε: If $k\varepsilon^2$ is the beginning of an excursion of β^ε, then T_k^ε is the hitting time of 2ε by the corresponding excursion of β. As in Section 1, consider for every $\varepsilon \in (0,1]$ an integer $N_\varepsilon \geq 1$ and assume that the family $(\varepsilon N_\varepsilon, \varepsilon \in (0,1])$ is bounded and that $\varepsilon N_\varepsilon$ converges to $a \geq 0$ as $\varepsilon \to 0$ (this follows from (1.1) with $a = \langle \mu, 1 \rangle$). Let τ^ε denote the N_ε-th return of β^ε to 0. From the previous

observations, (2.1) and the standard approximation of Brownian local times by upcrossing numbers, we have

$$\lim_{\varepsilon\to 0} \tau^\varepsilon = \tau_a, \text{ a.s.}$$

We will write $\tau = \tau_a$ for simplicity.

On the time interval $[0,\tau^\varepsilon]$, the process β^ε makes N_ε independent excursions away from 0. These excursions can be viewed as representing the genealogical structure of the branching particle system introduced in Section 1. The set of edges, denoted by $\mathcal{T}_\varepsilon$, is then a random subset of $\{1,\ldots,N_\varepsilon\}\times\mathbf{U}$ and conditionally on $\mathcal{T}_\varepsilon$, the corresponding lengths are independent exponentials with mean ε. The function $(\beta^\varepsilon_s, s\in[0,\tau^\varepsilon])$ can be reconstructed from this collection of marked trees as shown by Fig. 1. Notice that for this reconstruction to be possible, it is essential to order the trees and the different edges of every single tree.

2.3 Discrete and continuous local times

One reason for considering the processes β^ε comes from their relation with the upcrossing numbers of β. We first define the (discrete) local times of β^ε. For every $x\ge 0$ and $s\ge 0$, we define

$$L^{\varepsilon,x}_s = \varepsilon\,\mathrm{Card}\{r\in[0,s) : \beta^\varepsilon_r = x \text{ and } \beta^\varepsilon_u > x \text{ for } u\in(r,r+\delta], \text{ for some } \delta>0\}.$$

In other words, $\varepsilon^{-1}L^{\varepsilon,x}_s$ is the number of upcrossings of β^ε above level x before time s.

Let $M^\varepsilon_s(x)$ denote the number of upcrossings of β from x to $x+2\varepsilon$ completed before time s. More precisely, $M^\varepsilon_s(x)$ is the number of pairs (u,v) with $0\le u<v<s$, such that $\beta_u = x$, $\beta_r > x$ for every $r\in(u,v)$ and $v=\inf\{r>u:\beta_r > x+2\varepsilon\}$.

Then, a.s. for every $x\ge 0$ and every integer $k\ge 1$, we have

$$L^{\varepsilon,x}_{(2k-1)\varepsilon^2} = L^{\varepsilon,x}_{2k\varepsilon^2} = \varepsilon M^\varepsilon_{T^\varepsilon_{2k}}(x) = \varepsilon M^\varepsilon_{T^\varepsilon_{2k-1}}(x). \tag{2.4}$$

This identity is easily verified by induction on k (the sequence of stopping times (T^ε_k) was designed for this property to hold). See also Proposition 7 of [L1].

Lemma 2.1. *We have a.s.*

$$\lim_{\varepsilon\to 0}\Big(\sup_{s\ge 0}\sup_{x\ge 0}|L^{\varepsilon,x}_{s\wedge\tau^\varepsilon} - L^x_{s\wedge\tau}|\Big) = 0.$$

Proof. We first observe that a.s.

$$\lim_{\varepsilon\to 0}\Big(\sup_{s\ge 0}\sup_{x\ge 0}|\varepsilon M^\varepsilon_{s\wedge\tau^\varepsilon}(x) - L^x_{s\wedge\tau}|\Big) = 0. \tag{2.5}$$

For a fixed value of x, this is nothing but the classical approximation of Brownian local time by upcrossing numbers, and excursion theory provides precise estimates for the rate of convergence. Using these estimates and monotonicity properties, it is then an easy task to prove (2.5), i.e., the uniform version of the claim.

The statement of the lemma is now a simple consequence of (2.1), (2.4) and (2.5). $\square$

Remark. As an immediate consequence of Lemma 2.1 and the joint continuity of Brownian local times, we get that

$$\lim_{\varepsilon,\delta\to 0}\Big(\sup_{s\geq 0}\sup_{\substack{x,x'\geq 0\\ |x-x'|\leq\delta}}|L^{\varepsilon,x}_{s\wedge\tau^\varepsilon}-L^{\varepsilon,x'}_{s\wedge\tau^\varepsilon}|\Big)=0,\quad \text{a.s.}$$

Later, we will consider for every $\varepsilon\in(0,1]$ a process $\widetilde{\beta}^\varepsilon$ with the same distribution as β^ε. If $\widetilde{L}^{\varepsilon,x}_s$ denote the discrete local times of $\widetilde{\beta}^\varepsilon$, the last convergence still holds in probability when $L^{\varepsilon,x}_s$ is replaced by $\widetilde{L}^{\varepsilon,x}_s$ (and τ^ε by $\widetilde{\tau}^\varepsilon$, with an obvious notation).

2.4 Branching particle systems and discrete snakes.

We now consider the branching particle system of Section 1, starting with N_ε particles located respectively at $x^\varepsilon_1,x^\varepsilon_2,\ldots,x^\varepsilon_{N_\varepsilon}$. We may and will assume that the genealogy of the descendants of particle k (present at x^ε_k at time 0) is given by the tree associated with the k-th excursion of β^ε (cf subsection 2.2). We will refer to this system as the ε-system of branching Brownian motions.

For our purposes, it will be convenient to view the collection of paths traced by the branching particles as the range of a path-valued process called the discrete snake.

By definition, a stopped path is a continuous mapping $w:[0,\zeta]\longrightarrow\mathbb{R}$, where $\zeta=\zeta_w\geq 0$ is called the "lifetime" of w (it is convenient to talk about the "lifetime" of a path although for technical reasons the path is stopped rather than killed). Let $\mathcal{W}$ be the set of all stopped paths. Then $\mathcal{W}$ is a separable complete metric space for the distance

$$d(w,w')=|\zeta_w-\zeta_{w'}|+\sup_{t\geq 0}|w(t\wedge\zeta_w)-w'(t\wedge\zeta_{w'})|.$$

For any $x\in\mathbb{R}$, we write $\underline{x}$ for the trivial path such that $\zeta_{\underline{x}}=0$ and $\underline{x}(0)=x$.

With every $s\in[0,\tau^\varepsilon]$ we now associate a stopped path $W^\varepsilon_s\in\mathcal{W}$ with lifetime β^ε_s. If $s\in[0,\tau^\varepsilon)\cap\varepsilon^2\mathbb{N}$ and $\beta^\varepsilon_s=0$, then s is the starting time of the k-th excursion of β^ε away from 0, for some $k\in\{1,\ldots,N_\varepsilon\}$. We then set $W^\varepsilon_s=\underline{x}^\varepsilon_k$. For definiteness, we also set $W^\varepsilon_{\tau^\varepsilon}=\underline{x}^\varepsilon_{N_\varepsilon}$. If $s\in[0,\tau^\varepsilon)\cap\varepsilon^2\mathbb{N}$ but $\beta^\varepsilon_s>0$, we can associate with s a unique edge of the k-th tree, k being the number of the excursion straddling s. We then let W^ε_s be the historical path of the particle in the system of branching Brownian motions that corresponds to this

edge. Notice that the death time of this particle is β^ε_s, and thus $\zeta_{W^\varepsilon_s} = \beta^\varepsilon_s$. Finally if $s \in [0,\tau^\varepsilon]$ but $s \notin \varepsilon^2\mathbb{N}$, we find an integer j such that $j\varepsilon^2 < s < (j+1)\varepsilon^2$, and let $l = j$ if $\beta^\varepsilon_{j\varepsilon^2} > \beta^\varepsilon_{(j+1)\varepsilon^2}$, but $l = j+1$ if $\beta^\varepsilon_{j\varepsilon^2} \le \beta^\varepsilon_{(j+1)\varepsilon^2}$. Then we let W^ε_s be the path $W^\varepsilon_{l\varepsilon^2}$ stopped at time β^ε_s.

It is easy to see that conditionally on $(\beta^\varepsilon_s, s \ge 0)$ the process $(W^\varepsilon_{k\varepsilon^2}, 0 \le k \le \tau^\varepsilon/\varepsilon^2)$ is Markovian. To describe its conditional distribution, let $k \in \{0,\ldots,\tau^\varepsilon/\varepsilon^2\}$ and suppose that $\beta^\varepsilon_{(k+1)\varepsilon^2} > 0$ (otherwise $W^\varepsilon_{(k+1)\varepsilon^2} = \underline{x}^\varepsilon_j$, if $(k+1)\varepsilon^2$ is the starting point of the j-th excursion of β^ε). If $\beta^\varepsilon_{(k+1)\varepsilon^2} \le \beta^\varepsilon_{k\varepsilon^2}$ (which occurs if k is odd) then $W^\varepsilon_{(k+1)\varepsilon^2}$ is simply the restriction of $W^\varepsilon_{k\varepsilon^2}$ to $[0,\beta^\varepsilon_{(k+1)\varepsilon^2}]$. On the other hand, if $\beta^\varepsilon_{(k+1)\varepsilon^2} > \beta^\varepsilon_{k\varepsilon^2}$, then $W^\varepsilon_{(k+1)\varepsilon^2}$ is obtained from $W^\varepsilon_{k\varepsilon^2}$ by "adding at the tip of $W^\varepsilon_{k\varepsilon^2}$" a Brownian path of length $\beta^\varepsilon_{(k+1)\varepsilon^2} - \beta^\varepsilon_{k\varepsilon^2}$ independent of $(W^\varepsilon_{j\varepsilon^2}, j \le k)$.

The following *snake property* is a consequence of the definition of W^ε_s: If $s < s'$ and s and s' belong to the same (open) excursion interval of β^ε away from 0, then $W^\varepsilon_s(t) = W^\varepsilon_{s'}(t)$ for every $t \in [0, \inf_{u\in[s,s']} \beta^\varepsilon_u]$.

2.5 Convergence to super-Brownian motion

As in Section 1, we let X^ε_t be ε times the sum of the point masses at the positions of the particles alive at time t in the ε-system. This is equivalent to writing

$$X^\varepsilon_t = \int_0^{\tau_\varepsilon} dL^{\varepsilon,t}_s \, \delta_{W^\varepsilon_s(t)}.$$

To justify this formula, recall the correspondence between excursions and trees described in Subsection 2.2 and note that each upcrossing time s of β^ε above level t corresponds to one particle alive at time t, whose position is $W^\varepsilon_s(t)$. Similarly, the historical process Y^ε_t is

$$Y^\varepsilon_t = \int_0^{\tau_\varepsilon} dL^{\varepsilon,t}_s \, \delta_{W^\varepsilon_s}.$$

Recall our assumptions (1.1) and (1.2) and the convergence result in (1.3). We next prove a result about the uniform modulus of continuity for the paths W^ε_s. For convenience, we make the convention that $W^\varepsilon_s(t) = W^\varepsilon_s(\beta^\varepsilon_s)$ when $t > \beta^\varepsilon_s$.

Lemma 2.2. *Let $\eta \in (0,\frac{1}{2})$. Then,*

$$\lim_{\delta\downarrow 0}\Big(\inf_{\varepsilon\in(0,1]} P\big[|W^\varepsilon_s(t+r) - W^\varepsilon_s(t)| \le r^{\frac{1}{2}-\eta},\ \textit{for every } t \ge 0,\ r\in[0,\delta],\ s\in[0,\tau^\varepsilon]\big]\Big) = 1.$$

Remark. This is of course reminiscent of the uniform modulus of continuity for historical paths of super-Brownian motion. This lemma is therefore very close to the results of [DIP] and [DP], which however use different approximations.

Proof. Obviously it is enough to treat the case when $x_1^\varepsilon = \cdots = x_{N_\varepsilon}^\varepsilon = 0$ for every ε. We then use an embedding technique that will also play an important role later. Let $(W_s, s \geq 0)$ be the Brownian snake of [L2] driven by the process $(\beta_s, s \geq 0)$ and with starting point $\underline{0}$. Recall that this is a continuous Markov process with values in $\mathcal{W}_0 := \{w \in \mathcal{W} : w(0) = 0\}$, whose law is characterized by the following properties:

• For every $s \geq 0$, the path W_s has lifetime β_s.

• Conditionally on $(\beta_s, s \geq 0)$, the process $(W_s, s \geq 0)$ is time-inhomogeneous Markov, and its transition kernels are characterized as follows. If $s < s'$, we have $W_{s'}(t) = W_s(t)$ for every $t \leq m(s,s') := \inf_{[s,s']} \beta_r$, and $(W_{s'}(m(s,s') + r) - W_{s'}(m(s,s')), 0 \leq r \leq \beta_{s'} - m(s,s'))$ is a Brownian path independent of W_s.

Now, for every $\varepsilon \in (0,1]$, we may assume that the spatial motions of the particles are chosen in such a way that, for every $\varepsilon > 0$ and every $k \in \{0, 1, \ldots, \tau^\varepsilon/\varepsilon^2\}$,

$$\begin{aligned} W^\varepsilon_{k\varepsilon^2} &= W_{T_k^\varepsilon} \text{ if } k \text{ is odd,} \\ W^\varepsilon_{k\varepsilon^2} &= W_{T_k^\varepsilon} \mid [0, \beta_{T_k^\varepsilon} - 2\varepsilon] \text{ if } k \text{ is even,} \end{aligned} \tag{2.6}$$

where the notation $W_{T_k^\varepsilon} \mid [0, \beta_{T_k^\varepsilon} - 2\varepsilon]$ means that the path $W_{T_k^\varepsilon}$ is restricted to the interval $[0, \beta_{T_k^\varepsilon} - 2\varepsilon] = [0, \beta^\varepsilon_{k\varepsilon^2}]$. In fact, it is immediate to verify that the process $(W^\varepsilon_{k\varepsilon^2}, 0 \leq k \leq \tau^\varepsilon/\varepsilon^2)$ defined by (2.6) has (conditionally on β^ε) the distribution described at the end of Subsection 2.4.

Note that the family $(\tau^\varepsilon, \varepsilon \in (0,1])$ is bounded a.s. Then the proof of Lemma 2.2 reduces to checking that, for every $K > 0$,

$$\lim_{\delta \downarrow 0} P\big[|W_s(t+r) - W_s(t)| \leq r^{\frac{1}{2}-\eta}, \text{ for every } t \geq 0, r \in [0,\delta], s \in [0,K]\big] = 1. \tag{2.7}$$

This can be easily done using Borel-Cantelli type arguments. Alternatively, we may also use the relations between super-Brownian motion and the Brownian snake [L2], and the uniform modulus of continuity of [DP]. $\square$

The graph $\mathcal{G}^\varepsilon$ of the ε-system of branching particles is defined by

$$\mathcal{G}^\varepsilon = \mathrm{cl}\Big(\bigcup_{t \geq 0} \{t\} \times \operatorname{supp} X_t^\varepsilon\Big) = \{W_s^\varepsilon(t) : s \in [0, \tau^\varepsilon], 0 \leq t \leq \beta_s^\varepsilon\}.$$

We are interested in weak convergence of $\mathcal{G}^\varepsilon$ towards the graph $\mathcal{G}$ of X, which we define as

$$\mathcal{G} = \mathrm{cl}\Big(\bigcup_{t \geq 0} \{t\} \times \operatorname{supp} X_t\Big).$$

We view both $\mathcal{G}^\varepsilon$ and $\mathcal{G}$ as random elements of the space of all compact subsets of $\mathbb{R}_+ \times \mathbb{R}$, which is equipped with the Hausdorff metric.

Lemma 2.3. *We have the joint convergence*

$$\big((X_t^\varepsilon, t \geq 0), \mathcal{G}^\varepsilon\big) \xrightarrow[\varepsilon\to 0]{\text{(d)}} \big((X_t, t \geq 0), \mathcal{G}\big).$$

Proof. We first consider the case when $x_1^\varepsilon = \cdots = x_{N_\varepsilon}^\varepsilon = 0$ for every ε. Then we can suppose that the processes $(W_s^\varepsilon, s \in [0, \tau^\varepsilon])$ are constructed via the embedding technique described in the preceding proof. From (2.1) and (2.6), we get

$$(W_{s\wedge\tau^\varepsilon}^\varepsilon, s \geq 0) \xrightarrow[\varepsilon\to 0]{\text{(a.s.)}} (W_{s\wedge\tau}, s \geq 0) \tag{2.8}$$

in the sense of uniform convergence. Using Lemma 2.1, we get

$$X_t^\varepsilon = \int_0^{\tau^\varepsilon} dL_s^{\varepsilon,t}\, \delta_{W_s^\varepsilon(t)} \xrightarrow[\varepsilon\to 0]{\text{(a.s.)}} \int_0^\tau dL_s^t\, \delta_{W_s(t)} = X_t$$

uniformly in t. (The formula for X_t is the Brownian snake representation of super-Brownian motion, see [L2].) Furthermore, the convergence (2.8) also implies that

$$\mathcal{G}^\varepsilon = \{W_s^\varepsilon(t) : s \leq \tau^\varepsilon, t \leq \beta_s^\varepsilon\} \xrightarrow[\varepsilon\to 0]{\text{(a.s.)}} \{W_s(t) : s \leq \tau, t \leq \beta_s\},$$

and the limit is easily identified with the graph $\mathcal{G}$ of X. Therefore we get the statement of the lemma in the special case $x_1^\varepsilon = \cdots = x_{N_\varepsilon}^\varepsilon = 0$.

Before proceeding to the general case, let us make one more observation. Fix $\delta > 0$ and write $(V_s^\varepsilon, s \geq 0)$ for a process distributed as an excursion of W^ε away from $\underline{0}$ conditioned to have height greater than δ. (Alternatively, $(V_s^\varepsilon, s \geq 0)$ codes the historical paths of the ε-system starting with one particle at the origin and conditioned to be non-extinct at time δ.) It follows from the convergence (2.8) that we have also

$$(V_s^\varepsilon, s \geq 0) \xrightarrow[\varepsilon\to 0]{\text{(d)}} (V_s, s \geq 0),$$

where the limiting process is an excursion of W conditioned to have height greater than δ. As in the first part of the proof, it follows that the graphs of V^ε (defined analogously to $\mathcal{G}^\varepsilon$) also converge in distribution towards the graph of V. Furthermore, this convergence holds jointly with that of the measure-valued processes $\mathcal{X}_t^\varepsilon$ associated with V^ε in the same way as X_t^ε was associated with W^ε.

Let us consider now the general case. Because of Lemma 2.2 and assumption (1.2), it is enough to prove that for any fixed $\delta > 0$, $\mathcal{G}_\varepsilon \cap ([\delta, \infty) \times \mathbb{R})$ converges in distribution to $\mathcal{G} \cap ([\delta, \infty) \times \mathbb{R})$ (and that this convergence holds jointly with that of X^ε). Let A_ε stand for the set of indices $j \in \{1, \ldots, N_\varepsilon\}$ such that the j-th excursion of β^ε has a height

greater than δ. Note that the events $\{j \in A_\varepsilon\}$ are independent with the same probability $2\varepsilon(2\varepsilon+\delta)^{-1}$. It follows that the random measure

$$\sum_{j \in A_\varepsilon} \delta_{x_j^\varepsilon}$$

converges weakly to a Poisson measure with intensity $\frac{2}{\delta}\mu$. Note that, conditionally on A_ε, $\mathcal{G}_\varepsilon \cap [\delta,\infty) \times \mathbb{R}$ has the same distribution as

$$\bigcup_{j \in A_\varepsilon} \big(((0, x_j^\varepsilon) + \mathcal{G}_{j,\varepsilon}) \cap [\delta,\infty) \times \mathbb{R}\big)$$

where $\mathcal{G}_{j,\varepsilon}$ are independent copies of the graph of V^ε. If follows that the random sets $\mathcal{G}_\varepsilon \cap [\delta,\infty) \times \mathbb{R}$ converge in distribution to

$$\bigcup_{j \in J} \big(((0, x_j) + \mathcal{G}_{(j)}) \cap [\delta,\infty) \times \mathbb{R}\big),$$

where $\sum_{j \in J} \delta_{x_j}$ is a Poisson point measure on $\mathbb{R}$ with intensity $\frac{2}{\delta}\mu$, and, conditionally on this random measure, the random sets $\mathcal{G}_{(j)}$ are independent and distributed according to the law of the graph of V. The canonical representation of superprocesses allows us to identify this limiting distribution with that of $\mathcal{G} \cap ([\delta,\infty) \times \mathbb{R})$. Furthermore, using the joint convergence of $(V^\varepsilon, \mathcal{X}^\varepsilon)$, it is easy to verify that the convergence holds jointly with that of X_ε. $\square$

3. Branching particle systems with reflection

3.1 Reflection for deterministic paths

The purpose of this section is to explain, first in a deterministic setting, the construction of reflected systems. We consider a deterministic branching particle system in $\mathbb{R}$ analogous to the ones considered above. At time 0, we have N particles located at $x_1, \ldots, x_N$. Each particle moves in $\mathbb{R}$ and gives birth at its death to 0 or 2 new particles. As in Subsection 2.2, denote by $\mathcal{T}$ the genealogical forest of the population, which is a subset of $\{1,\ldots,N\} \times \mathbf{U}$. Each element $v = (k,u)$ in $\mathcal{T}$ corresponds to a particle with birth time ξ_v and death time ζ_v (as in Section 2, we could alternatively consider the life durations $\ell_v := \zeta_v - \xi_v$ but in this subsection and the next one it is more convenient to deal with the birth and death times). The spatial motion of v is a continuous function $f_v : [\xi_v, \zeta_v] \longrightarrow \mathbb{R}$ and $f_{v'}(\xi_{v'}) = f_v(\zeta_v)$ if v' is a child of v (then $\xi_{v'} = \zeta_v$). The historical path of v is the continuous function $w_v : [0,\zeta_v] \longrightarrow \mathbb{R}$ such that, for every $t \in [0,\zeta_v)$, $w_v(t)$ is the position at time t of the ancestor of v alive at that time.

We assume that the death times ζ_v, $v \in \mathcal{T}$ are all distinct, that the system becomes extinct after a finite number of generations and that when a particle dies there is no other particle at the same location: For every $v \in \mathcal{T}$, $f_v(\zeta_v) \neq f_{v'}(\zeta_v)$ for every $v' \in \mathcal{T}$ such that $\xi_{v'} \leq \zeta_v < \zeta_{v'}$.

We turn to the construction of the reflected system. This system is such that the number and positions of the particles alive at every time t are the same as in the original system (thus each death time for the reflected system is also a death time for the reflected system). However the genealogical forest $\widetilde{\mathcal{T}}$ will be different, as will be the spatial motions $\widetilde{f}_u$, $u \in \widetilde{\mathcal{T}}$ or the birth and death times $\widetilde{\xi}_u$, $\widetilde{\zeta}_u$, $u \in \widetilde{\mathcal{T}}$.

Set $R_0 = 0$ and denote by $R_1 < R_2 < \cdots < R_M$ the successive death times in the original system. For every $k \in \{1, \ldots, M\}$, let $\mathcal{T}_{(k)}$ be the set of (labels of) particles that are alive on the interval $[R_{k-1}, R_k)$. We use induction on k to define sets $\widetilde{\mathcal{T}}_{(k)}$, which will represent the particles alive on the interval $[R_{k-1}, R_k)$ in the reflected system, and the corresponding spatial motions.

To begin with, we have $\widetilde{\mathcal{T}}_{(1)} = \{1, \ldots, N\}$, and we define $\widetilde{f}_j(t)$ for every $t \in [0, R_1]$ and every $j \in \widetilde{\mathcal{T}}_{(1)}$ by requiring $(\widetilde{f}_1(t), \ldots, \widetilde{f}_N(t))$ to be the increasing rearrangement of $(f_1(t), \ldots, f_N(t))$. Note that the mappings $\widetilde{f}_1, \ldots, \widetilde{f}_N$ are continuous.

Suppose that for some $k \in \{1, \ldots, M-1\}$, we have defined $\widetilde{\mathcal{T}}_{(k)}$ and the corresponding paths $(\widetilde{f}_u(t), t \in [R_{k-1}, R_k])$, for $u \in \widetilde{\mathcal{T}}_{(k)}$, in such a way that $\mathrm{Card}\, \widetilde{\mathcal{T}}_{(k)} = \mathrm{Card}\, \mathcal{T}_{(k)}$, and, for every $t \in [R_{k-1}, R_k]$:

- The mapping $\widetilde{\mathcal{T}}_{(k)} \ni u \to \widetilde{f}_u(t)$ is increasing with respect to the lexicographical order on $\widetilde{\mathcal{T}}_{(k)}$.

- The values of $\widetilde{f}_u(t)$ for $u \in \widetilde{\mathcal{T}}_{(k)}$ (counted with their multiplicities) are the same as those of $f_u(t)$ for $u \in \mathcal{T}_{(k)}$.

By definition, one of the particles in $\mathcal{T}_{(k)}$, say $u_{(k)}$, dies at time R_k. Then there is exactly one $\widetilde{u}_{(k)} \in \widetilde{\mathcal{T}}_{(k)}$ such that $\widetilde{f}_{\widetilde{u}_{(k)}}(R_k) = f_{u_{(k)}}(R_k)$. We set

$$\widetilde{\mathcal{T}}_{(k+1)} = \big(\widetilde{\mathcal{T}}_{(k)} \backslash \{\widetilde{u}_{(k)}\}\big) \cup \{\widetilde{u}_{(k)}1, \widetilde{u}_{(k)}2\}$$

if $u_{(k)}$ has two children in the original system, and

$$\widetilde{\mathcal{T}}_{(k+1)} = \widetilde{\mathcal{T}}_{(k)} \backslash \{\widetilde{u}_{(k)}\}$$

if not. Furthermore, let $u_1^{k+1}, \ldots, u_{N_{k+1}}^{k+1}$ be the elements of $\widetilde{\mathcal{T}}_{(k+1)}$ listed in lexicographical order. We define $\widetilde{f}_u(t)$ for every $t \in [R_k, R_{k+1}]$ and every $u \in \widetilde{\mathcal{T}}_{(k+1)}$ by requiring that $(\widetilde{f}_{u_1^{k+1}}(t), \ldots, \widetilde{f}_{u_{N_{k+1}}^{k+1}}(t))$ is the increasing rearrangement of $(f_u(t), u \in \mathcal{T}_{(k+1)})$. Notice that when $u \in \widetilde{\mathcal{T}}_{(k)} \cap \widetilde{\mathcal{T}}_{(k+1)}$ the definition of $\widetilde{f}_u(R_k)$ is consistent with the previous step.

Finally, the genealogical forest of the reflected system is

$$\widetilde{\mathcal{T}} = \bigcup_{k=1}^{M} \widetilde{\mathcal{T}}_{(k)}.$$

The birth and death times $\widetilde{\xi}_u, \widetilde{\zeta}_u$ as well as the (continuous) spatial motions $\widetilde{f}_u$ in the reflected system are defined by the requirement of consistency with the construction of $\widetilde{\mathcal{T}}_{(k)}$'s. Note the two fundamental properties:

• At each time $t \geq 0$, the positions of the particles (counted with their multiplicities) are the same in the original and the reflected system.

• If $u, v \in \widetilde{\mathcal{T}}$ with $u \prec v$ ($\prec$ denotes the lexicographical order) then $f_u(t) \leq f_v(t)$ for every $t \in [\widetilde{\xi}_u, \widetilde{\zeta}_u] \cap [\widetilde{\xi}_v, \widetilde{\zeta}_v]$.

Historical paths $\widetilde{w}_u$, $u \in \widetilde{\mathcal{T}}$ for the reflected system are defined in a way analogous to the original one. If $u, v \in \widetilde{\mathcal{T}}$ and $u \prec v$ then $\widetilde{w}_u(t) \leq \widetilde{w}_v(t)$ for every $t \in [0, \widetilde{\zeta}_u \wedge \widetilde{\zeta}_v]$.

3.2 A technical lemma

Let $M \in \{1, \cdots, N\}$, and consider a branching system consisting only of the particles labeled $1, \ldots, M$ at time 0 and their descendants. The new genealogy is described by the forest

$$\mathcal{T}' := \mathcal{T} \cap (\{1, \ldots, M\} \times U).$$

From this new branching particle system, we can construct a reflected system by the procedure described in Subsection 3.1. We denote by $\widetilde{\mathcal{T}}'$ the genealogical forest for this new reflected system, and by $\widetilde{w}'_v$, $v \in \widetilde{\mathcal{T}}'$ the associated historical paths. In general, the historical paths $\widetilde{w}'_v$ will be very different from those obtained by reflecting the original system. Under special assumptions however, we can say that some of the paths $\widetilde{w}'_v$ will also be (reflected) historical paths in the original system.

Lemma 3.1. *Let $t > 0$ and let I be a bounded interval in $\mathbb{R}$. Suppose that $w_v(r) \notin I$ for every $v \in \mathcal{T} \backslash \mathcal{T}'$ and $r \in [0, t]$. If $v \in \widetilde{\mathcal{T}}$ is such that $\widetilde{\zeta}_v \geq t$ and $\widetilde{w}_v(r) \in I$ for every $r \in [0, t]$, then there exists $v' \in \widetilde{\mathcal{T}}'$ such that $\widetilde{\zeta}'_{v'} \geq t$ and $\widetilde{w}'_{v'}(r) = \widetilde{w}_v(r)$ for every $r \in [0, t]$. The converse also holds: If $v' \in \widetilde{\mathcal{T}}'$ is such that $\widetilde{\zeta}'_{v'} \geq t$ and $\widetilde{w}'_{v'}(r) \in I$ for every $r \in [0, t]$, then there exists $v \in \widetilde{\mathcal{T}}$ such that $\widetilde{\zeta}_v \geq t$ and $\widetilde{w}_v(r) = \widetilde{w}'_{v'}(r)$ for every $r \in [0, t]$.*

In other words, the first assertion means that the path $\widetilde{w}_v$, or rather its restriction to $[0, t]$, will still be a historical path for the new reflected system. We leave an easy proof of the lemma to the reader.

3.3 Reflected branching particle systems

For every $\varepsilon \in (0,1]$, we can apply the construction of Subsection 3.1 to the ε-system of branching Brownian motions. Note that the assumptions that we imposed on the deterministic system hold with probability one for this random system. We write $\mathcal{T}_\varepsilon$ for the genealogical forest of the ε-system, and $(\ell^\varepsilon_u, u \in \mathcal{T}_\varepsilon)$ for the lifetimes of particles. The notation $\widetilde{\mathcal{T}}_\varepsilon$ and $(\widetilde{\ell}^\varepsilon_u, u \in \widetilde{\mathcal{T}}_\varepsilon)$ has a similar meaning for the corresponding reflected system, which we call the ε-reflected system. Observe that $(\mathcal{T}_\varepsilon, (\ell^\varepsilon_u, u \in \mathcal{T}_\varepsilon))$ and $(\widetilde{\mathcal{T}}_\varepsilon, (\widetilde{\ell}^\varepsilon_u, u \in \widetilde{\mathcal{T}}_\varepsilon))$ have the same distribution. This is so because the spatial motions and branching structure for the ε-system of branching Brownian motions are independent (a tedious rigorous justification could be given, but we feel that the result is sufficiently obvious to allow us to omit it). Furthermore, $\mathrm{Card}\,\widetilde{\mathcal{T}}_\varepsilon = \mathrm{Card}\,\mathcal{T}_\varepsilon$.

We noticed at the end of Subsection 2.2 that the process $(\beta^\varepsilon_s, s \in [0,\tau^\varepsilon])$ can be reconstructed as a measurable function of the marked trees $(\mathcal{T}_\varepsilon, (\ell^\varepsilon_u, u \in \mathcal{T}_\varepsilon))$. Hence, we can also code the branching structure of the ε-reflected system by a random process $(\widetilde{\beta}^\varepsilon_s, s \in [0,\widetilde{\tau}^\varepsilon])$ which has the same distribution as $(\beta^\varepsilon_s, s \in [0,\tau^\varepsilon])$. The fact that $\mathrm{Card}\,\widetilde{\mathcal{T}}_\varepsilon = \mathrm{Card}\,\mathcal{T}_\varepsilon$ implies that the time τ_ε is also the end of the N_ε-th excursion of $\widetilde{\beta}^\varepsilon$ away from 0, and thus $\widetilde{\tau}^\varepsilon = \tau^\varepsilon$. The discrete local times of $\widetilde{\beta}^\varepsilon$ (cf. Subsection 2.3) are denoted by $(\widetilde{L}^{\varepsilon,x}_s, x \in \mathbb{R}_+, s \in [0,\tau^\varepsilon])$.

Finally, we can code the historical paths of the ε-reflected system by a discrete snake $(\widetilde{W}^\varepsilon_s, s \in [0,\tau^\varepsilon])$ in a way analogous to what we did in Subsection 2.4. Recall that we assume $x^\varepsilon_1 \leq \cdots \leq x^\varepsilon_{N_\varepsilon}$. As in Section 2, if $s \in \varepsilon^2\mathbb{N} \cap [0,\tau^\varepsilon)$ and $\widetilde{\beta}^\varepsilon_s = 0$, we set $\widetilde{W}^\varepsilon_s = \underline{x}^\varepsilon_k$ if s is the beginning of the k-th excursion of $\widetilde{\beta}^\varepsilon$ away from 0 (and $\widetilde{W}^\varepsilon_{\tau_\varepsilon} = \underline{x}^\varepsilon_{N_\varepsilon}$). Otherwise, if $s \in \varepsilon^2\mathbb{N} \cap [0,\tau^\varepsilon)$ and $\widetilde{\beta}^\varepsilon_s > 0$, then $(s, \widetilde{\beta}^\varepsilon_s)$ can be associated with a unique edge u of the forest $\widetilde{\mathcal{T}}_\varepsilon$, and we let $\widetilde{W}^\varepsilon_s$ be equal to $\widetilde{w}^\varepsilon_u$, the historical path of u. If $s \notin \varepsilon^2\mathbb{N}$, we use the same interpolation as in Section 2. A fundamentally important property of the process $(\widetilde{W}^\varepsilon_s, s \in [0,\tau^\varepsilon])$, from the point of view of our project, is that for $s < s'$,

$$\widetilde{W}^\varepsilon_s(t) \leq \widetilde{W}^\varepsilon_{s'}(t), \qquad \forall\, t \in [0, \widetilde{\beta}^\varepsilon_s \wedge \widetilde{\beta}^\varepsilon_{s'}]. \tag{3.1}$$

This follows from our construction and the end of Subsection 3.1. As in the case of W^ε_s, we see that if $s < s'$, then

$$\widetilde{W}^\varepsilon_s(t) = \widetilde{W}^\varepsilon_{s'}(t), \qquad \forall\, t \in [0, \inf_{u\in[s,s']} \widetilde{\beta}^\varepsilon_u].$$

Because at every time the locations of particles are the same in the reflected system and in the original one, the random measure

$$\widetilde{X}^\varepsilon_t = \int_0^{\tau_\varepsilon} d\widetilde{L}^{\varepsilon,t}_s \, \delta_{\widetilde{W}^\varepsilon_s(t)}$$

coincides with X_t^ε. Things are however very different for the historical measure

$$\widetilde{Y}_t^\varepsilon = \int_0^{\tau_\varepsilon} d\widetilde{L}_s^{\varepsilon,t}\, \delta_{\widetilde{W}_s^\varepsilon}.$$

4. Tightness of the reflected system

4.1 Uniform continuity of the reflected paths

Our first goal is to derive an important uniform continuity property for the individual paths of the ε-reflected system (Theorem 4.1 below). From the intuitive point of view, reflected paths should have smaller oscillations than “free” paths and so this property seems to be a straightforward consequence of Lemma 2.2. However the intuition about the relationship between moduli of continuity of free and reflected paths is only correct as long as we do not have any deaths. To be specific consider p paths $w_{(1)}, \ldots, w_{(p)}$ all defined on the time interval $[0,1]$, and let $\widetilde{w}_{(1)}, \ldots, \widetilde{w}_{(p)}$ be the corresponding system of reflected paths. Then, if we assume that $|w_{(i)}(t) - w_{(i)}(t')| \le \varphi(|t-t'|)$ for every $i = 1, \ldots, p$ and $t, t' \in [0,1]$ and for some nondecreasing function φ, an easy argument shows that the same bound holds when the paths $w_{(i)}$ are replaced by $\widetilde{w}_{(i)}$.

It turns out that a similar assertion about moduli of continuity is false if paths may have different lifetimes. Fig. 2 shows a system of two paths. In the original system, the oscillations of paths over the intervals where they are defined are equal to $z_1 - y_1$ and $z_2 - y_2$. One of the paths in the reflected system goes from y_1 to z_2 and so has an oscillation larger than the oscillations of the original paths. In this article we consider Brownian particles which die at different times so we cannot use known estimates for the modulus of continuity of the original (non-reflecting) historical paths in a direct way. We will use them later in a different but quite essential way.

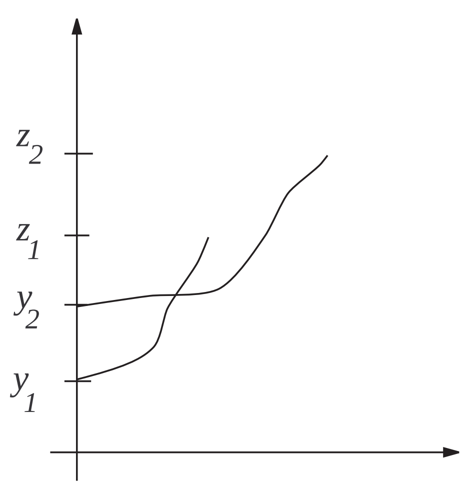

Figure 2.

Recall our notation $\widetilde{w}_u^\varepsilon$, $u \in \widetilde{\mathcal{T}}_\varepsilon$, for the historical paths of the ε-reflected system. By convention, $\widetilde{w}_u^\varepsilon(t) = \widetilde{w}_u^\varepsilon(\widetilde{\zeta}_u^\varepsilon)$ if $t \geq \widetilde{\zeta}_u^\varepsilon$.

Theorem 4.1. *For every $\eta > 0$,*

$$\lim_{\delta\to 0}\Big(\limsup_{\varepsilon\to 0} P\Big[\sup_{\substack{t,t'\geq 0\\ |t-t'|\leq\delta}}\ \sup_{u\in\widetilde{\mathcal{T}}_\varepsilon} |\widetilde{w}_u^\varepsilon(t) - \widetilde{w}_u^\varepsilon(t')| > \eta\Big]\Big) = 0.$$

Proof. Let $(\delta_{(p)}, \varepsilon_{(p)})$ be a sequence in $(0,1]^2$ converging to 0. We will prove that there exists a subsequence $(\delta'_{(p)}, \varepsilon'_{(p)})$ such that:

$$\lim_{p\to\infty}\Big(\sup_{\substack{t,t'\geq 0\\ |t-t'|\leq\delta'_{(p)}}}\ \sup_{u\in\widetilde{\mathcal{T}}_{\varepsilon'_{(p)}}} |\widetilde{w}_u^{\varepsilon'_{(p)}}(t) - \widetilde{w}_u^{\varepsilon'_{(p)}}(t')|\Big) = 0 \tag{4.1}$$

in probability. Clearly, the statement of Theorem 4.1 is a consequence of this fact.

We first explain how we choose the sequence $(\delta'_{(p)}, \varepsilon'_{(p)})$. By Lemma 2.3 and the Skorohod representation theorem ([EK] Theorem 3.1.8), we may, for every $p \geq 1$, replace the pair $(X^{\varepsilon_{(p)}}, \mathcal{G}_{\varepsilon_{(p)}})$ by a new pair with the same distribution (for which we keep the same notation), in such a way that

$$(X^{\varepsilon_{(p)}}, \mathcal{G}_{\varepsilon_{(p)}}) \xrightarrow[p\to\infty]{\text{(a.s.)}} (X, \mathcal{G}),$$

where X is a super-Brownian motion started at μ and $\mathcal{G}$ denotes its graph. Note that the genealogical forest $\widetilde{\mathcal{T}}_{\varepsilon_{(p)}}$, the process $\widetilde{\beta}^{\varepsilon_{(p)}}$ and the historical paths $\widetilde{w}_u^{\varepsilon_{(p)}}$, $u \in \widetilde{\mathcal{T}}_{\varepsilon_{(p)}}$, are reconstructed as measurable functions of the new process $X^{(\varepsilon_p)}$, and that it suffices to prove (4.1) for the new historical paths. As a consequence of the remark following Lemma 2.1, we have

$$\lim_{p\to\infty}\Big(\sup_{s\geq 0}\ \sup_{\substack{t,t'\geq 0\\ |t-t'|\leq\delta_{(p)}}} |\widetilde{L}^{\varepsilon_{(p)},t}_{s\wedge\tau^{\varepsilon_{(p)}}} - \widetilde{L}^{\varepsilon_{(p)},t'}_{s\wedge\tau^{\varepsilon_{(p)}}}|\Big) = 0 \tag{4.2}$$

in probability. We choose the subsequence $(\delta'_{(p)}, \varepsilon'_{(p)})$ so that the convergence (4.2) holds almost surely along this subsequence.

We will argue by contradiction to prove (4.1). If (4.1) does not hold, then on a set A of positive probability, we can find a number $\eta > 0$ and a (random) subsequence $p_k \uparrow \infty$ such that, if $\varepsilon_k := \varepsilon'_{(p_k)}$ and $\delta_k := \delta'_{(p_k)}$,

$$\sup_{\substack{t,t'\geq 0\\ |t-t'|\leq\delta_k}}\ \sup_{u\in\widetilde{\mathcal{T}}_{\varepsilon_k}} |\widetilde{w}_u^{\varepsilon_k}(t) - \widetilde{w}_u^{\varepsilon_k}(t')| > \eta. \tag{4.3}$$

From now on until the end of the proof, we will assume that the event A holds. By (4.3), for every $k \geq 1$, there exist $u_k \in \widetilde{\mathcal{T}}_{\varepsilon_k}$, $t_k, t'_k \geq 0$ with $|t_k - t'_k| \leq \delta_k$, such that

$$|\widetilde{w}^{\varepsilon_k}_{u_k}(t_k) - \widetilde{w}^{\varepsilon_k}_{u_k}(t'_k)| > \eta.$$

Clearly, we can assume that $t_k \leq t'_k \leq \widetilde{\zeta}^{\varepsilon_k}_{u_k}$.

Recall that the graphs $\mathcal{G}_{\varepsilon_{(p)}}$ converge to $\mathcal{G}$ in the Hausdorff metric. In particular, the set of all pairs $(t_k, \widetilde{w}^{\varepsilon_k}_{u_k}(t_k))$ and $(t'_k, \widetilde{w}^{\varepsilon_k}_{u_k}(t'_k))$ is relatively compact. By passing to a subsequence, if necessary, we may assume that $t_k, t'_k \longrightarrow t_\infty$, $\widetilde{w}^{\varepsilon_k}_{u_k}(t_k) \longrightarrow x_1$ and $\widetilde{w}^{\varepsilon_k}_{u_k}(t'_k) \longrightarrow x_2$ as $k \to \infty$. We have $|x_1 - x_2| \geq \eta$, and we take $x_2 > x_1$ for definiteness.

We also know that $\widetilde{X}^{\varepsilon(p)}_t = X^{\varepsilon(p)}_t$ converges to X_t a.s. as $p \to \infty$, uniformly on compact subsets of $\mathbb{R}_+$. Hence, both sequences $\widetilde{X}^{\varepsilon_k}_{t_k}$ and $\widetilde{X}^{\varepsilon_k}_{t'_k}$ converge to X_{t_∞}, and

$$\begin{aligned} &\limsup_{k\to\infty} \widetilde{X}^{\varepsilon_k}_{t_k}((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t_k)]) \leq X_{t_\infty}((-\infty, x_1])\,, \\ &\liminf_{k\to\infty} \widetilde{X}^{\varepsilon_k}_{t'_k}((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t'_k))) \geq X_{t_\infty}((-\infty, x_2))\,. \end{aligned} \tag{4.4}$$

We claim that

$$\liminf_{k\to\infty} \Big(\widetilde{X}^{\varepsilon_k}_{t_k}\big((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t_k)]\big) - \widetilde{X}^{\varepsilon_k}_{t'_k}\big((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t'_k))\big)\Big) \geq 0. \tag{4.5}$$

To see this, we use the discrete snake representation of Subsection 3.3. Write $s_k \in [0, \tau_{\varepsilon_k}) \cap \varepsilon^2\mathbb{N}$ for the time associated with the edge u_k of $\widetilde{\mathcal{T}}_{\varepsilon_k}$ in this representation. By construction, $\widetilde{w}^{\varepsilon_k}_{u_k} = \widetilde{W}^{\varepsilon_k}_{s_k}$, and (3.1) implies

$$\widetilde{X}^{\varepsilon_k}_{t_k}((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t_k)]) = \int_0^{\tau_{\varepsilon_k}} d\widetilde{L}^{\varepsilon_k, t_k}_s \, \mathbf{1}_{\{\widetilde{W}^{\varepsilon_k}_s(t_k) \leq \widetilde{W}^{\varepsilon_k}_{s_k}(t_k)\}} \geq \widetilde{L}^{\varepsilon_k, t_k}_{s_k}.$$

Similarly, we get

$$\widetilde{X}^{\varepsilon_k}_{t'_k}((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t'_k))) \leq \widetilde{L}^{\varepsilon_k, t'_k}_{s_k}.$$

Hence,

$$\widetilde{X}^{\varepsilon_k}_{t_k}((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t_k)]) - \widetilde{X}^{\varepsilon_k}_{t'_k}((-\infty, \widetilde{w}^{\varepsilon_k}_{u_k}(t'_k))) \geq \widetilde{L}^{\varepsilon_k, t_k}_{s_k} - \widetilde{L}^{\varepsilon_k, t'_k}_{s_k}. \tag{4.6}$$

On the other hand,

$$|\widetilde{L}^{\varepsilon_k, t_k}_{s_k} - \widetilde{L}^{\varepsilon_k, t'_k}_{s_k}| \leq \sup_{s \in [0, \tau_{\varepsilon_k}]} \sup_{\substack{t, t' \geq 0 \\ |t - t'| \leq \delta_k}} |\widetilde{L}^{\varepsilon_k, t}_s - \widetilde{L}^{\varepsilon_k, t'}_s|.$$

By the convergence in (4.2), which holds a.s. along the subsequence $(\delta'_{(p)}, \varepsilon'_{(p)})$, the right hand side tends to 0 as $k \to \infty$. This and (4.6) give the claim (4.5).

From (4.5) and (4.4), we get $X_{t_\infty}((-\infty,x_1]) \geq X_{t_\infty}((-\infty,x_2))$ and thus (recall that $x_1 < x_2$), $X_{t_\infty}((x_1,x_2)) = 0$. This a priori does not imply that $\{t_\infty\} \times (x_1,x_2) \cap \mathcal{G} = \emptyset$ as there could be a "local extinction" of X at time t_∞ in (x_1,x_2). However, by Theorem 1.4 of Perkins [P1], there can be at most one local extinction at a given time, so we can choose x_1' and x_2' with $x_1 < x_1' < x_2' < x_2$ such that $\{t_\infty\} \times [x_1',x_2'] \cap \mathcal{G} = \emptyset$. Since $\mathcal{G}$ is closed, we have also $[t_\infty - \delta, t_\infty + \delta] \times [x_1',x_2'] \cap \mathcal{G} = \emptyset$ for $\delta > 0$ sufficiently small. However, by construction, for k sufficiently large the paths $\widetilde{w}^{\varepsilon_k}_{u_k}$ and thus also the graph $\mathcal{G}_{\varepsilon_k}$ must intersect $[t_\infty - \delta, t_\infty + \delta] \times [x_1',x_2']$. This gives a contradiction since we know that $\mathcal{G}_{\varepsilon_k}$ converge to $\mathcal{G}$. This contradiction completes the proof of Theorem 4.1. $\square$

4.2 Tightness of reflected discrete snakes

From now on, we restrict our attention to values of ε belonging to a fixed sequence $\mathcal{E}$ decreasing to 0. For convenience, we extend the definition of the discrete snakes $\widetilde{W}^\varepsilon$ by taking $\widetilde{W}^\varepsilon_s = \widetilde{W}^\varepsilon_{\tau^\varepsilon} = \underline{x}^\varepsilon_{N_\varepsilon}$ (and thus $\widetilde{\beta}^\varepsilon_s = 0$) for $s > \tau^\varepsilon$.

Proposition 4.2. *The laws of the processes $\widetilde{W}^\varepsilon$, $\varepsilon \in \mathcal{E}$, are tight in the space of all probability measures on $\mathbf{D}([0,\infty),\mathcal{W})$. Furthermore, if $(\widetilde{W}_s, s \geq 0)$ is a weak limit point of this sequence of processes, we have the following properties.*

(i) *If $\widetilde{\beta}_s := \zeta_{\widetilde{W}_s}$, the process $(\widetilde{\beta}_s, s \geq 0)$ has the same distribution as $(\beta_{s\wedge\tau}, s \geq 0)$.*

(ii) *Almost surely for every $s \leq s'$ we have $\widetilde{W}_s(t) \leq \widetilde{W}_{s'}(t)$ for every $t \in [0, \widetilde{\beta}_s \wedge \widetilde{\beta}_{s'}]$.*

(iii) *The set of discontinuities of the mapping $s \to \widetilde{W}_s$ is contained in the zero set of $\widetilde{\beta}$. Furthermore, if $s < s'$ belong to the same connected component of the complement of the zero set, we have*

$$\widetilde{W}_s(t) = \widetilde{W}_{s'}(t) \qquad \text{for every } t \in [0, \inf_{r\in[s,s']} \widetilde{\beta}_r]\,.$$

Proof. The hard part of the proof is to show tightness. To this end we rely on the classical criteria (see e.g. Corollary 3.7.4 of [EK]). We first observe that the compact containment condition is a straightforward consequence of Theorem 4.1. In fact, if $\eta > 0$ is fixed, then for every integer $p \geq 1$, Theorem 4.1 and the construction of the discrete snake $\widetilde{W}^\varepsilon$ allow us to find $\delta_p > 0$ such that, for $\varepsilon \in \mathcal{E}$ small enough,

$$P\Big[\sup_{s\geq 0} \sup_{\substack{t,t'\geq 0\\ |t-t'\leq \delta_p}} |\widetilde{W}^\varepsilon_s(t) - \widetilde{W}^\varepsilon_s(t')| > 2^{-p}\Big] \leq \eta\, 2^{-p-1}. \tag{4.7}$$

(Here and later, we make the convention that $\widetilde{W}^\varepsilon_s(t) = \widetilde{W}^\varepsilon_s(\widetilde{\beta}^\varepsilon_s)$ for $t > \widetilde{\beta}^\varepsilon_s$.) It is easy to see that an even stronger assertion holds, namely, (4.7) is true for all $\varepsilon \in \mathcal{E}$; this can be achieved by taking δ_p even smaller if necessary—note that for any fixed value of ε we need

only consider a finite number of historical paths. Then let H be a compact subset of $\mathbb{R}_+$ containing $\operatorname{supp}\mu_\varepsilon$ for $\varepsilon \in \mathcal{E}$, and let $A > 0$ be a constant. The set

$$\begin{aligned} K := &\{w \in \mathcal{W} : w(0) \in H\,, \zeta_w \le A\,, \\ &\quad \text{and } |w(t) - w(t')| \le 2^{-p} \text{ for every } t, t' \in [0, \zeta_w] \text{ with } |t - t'| \le \delta_p \text{ and every } p \ge 1\} \end{aligned}$$

is compact, and it follows from (4.7) that

$$P[\widetilde{W}^\varepsilon_s \notin K \text{ for some } s \ge 0] < \eta$$

provided that A is chosen large enough.

Recall the definition of the distance d from Subsection 2.4. We set

$$\theta(\varepsilon, \delta) = \inf_{(s_i)} \left\{ \sup_i \sup_{s, s' \in [s_{i-1}, s_i)} d(W^\varepsilon_s, W^\varepsilon_{s'}) \right\},$$

where the infimum is over all finite sequences $0 = s_0 < s_1 < \cdots < s_{m-1} < \tau^\varepsilon \le s_m$ such that $\inf\{|s_i - s_{i-1}|; 1 \le i \le m\} \ge \delta\}$. As a direct application of Corollary 3.7.4 in [EK], the proof of tightness will be complete if we can verify that, for every $\eta > 0$, we can choose $\delta > 0$ sufficiently small so that

$$\limsup_{\mathcal{E} \ni \varepsilon \to 0} P[\theta(\varepsilon, \delta) > \eta] < \eta. \tag{4.8}$$

We now fix $\eta > 0$ and proceed to the proof of (4.8). As a consequence of Theorem 4.1, we can choose $\rho \in (0, \eta/5)$ so small that, for every $\varepsilon \in \mathcal{E}$,

$$P\Big[\sup_{s \ge 0} \sup_{\substack{t, t' \ge 0 \\ |t - t'| \le \rho}} |\widetilde{W}^\varepsilon_s(t) - \widetilde{W}^\varepsilon_s(t')| \le \frac{\eta}{5}\Big] \ge 1 - \frac{\eta}{5}. \tag{4.9}$$

Then, by the tightness of the laws of $\widetilde{\beta}^\varepsilon$ (cf (2.3)), we can choose $\kappa > 0$ small enough so that, for every $\varepsilon \in \mathcal{E}$,

$$P\Big[\sup_{\substack{s, s' \ge 0 \\ |s - s'| \le \kappa}} |\widetilde{\beta}^\varepsilon_s - \widetilde{\beta}^\varepsilon_{s'}| \le \rho\Big] \ge 1 - \frac{\eta}{5}. \tag{4.10}$$

We denote by E_ε the intersection of the events considered in (4.9) and (4.10), so that the probability of the complement of E_ε is bounded above by $2\eta/5$.

Set $\gamma = \eta/5$. Since μ is a finite measure with compact support, we can easily find an integer M_γ and a finite sequence of reals $y_1 < z_1 \le y_2 < z_2 \le \cdots \le y_{M_\gamma} < z_{M_\gamma}$, such that:

- $z_i - y_i < \gamma$ for every $i = 1, \ldots, M_\gamma$,
- $\displaystyle\bigcup_{i=1}^{M_\gamma} [y_i, z_i)$ contains a neighborhood of $\operatorname{supp}\mu$,

- $\mu(\{y_i\}) = \mu(\{z_i\}) = 0$, and $\mu([y_i, z_i)) > 0$ for every $i = 1, \dots M_\gamma$.

By the last condition, $a_\gamma := \inf\{\mu([y_i, z_i))\,,\, i = 1, \dots M_\gamma\} > 0$. Furthermore, if ε is small enough,

$$\operatorname{supp} \mu_\varepsilon \subset \bigcup_{i=1}^{M_\gamma} [y_i, z_i)$$

and

$$\operatorname{Card}\{j : x_j^\varepsilon \in [y_i, z_i)\} > \frac{a_\gamma}{2\varepsilon} \geq 1,$$

for every $i = 1, \dots, M_\gamma$. From now on, we assume that $\varepsilon \in \mathcal{E}$ is small enough so that the last two conditions hold, and we set

$$n_i^\varepsilon = \inf\{j : x_j^\varepsilon \in [y_i, z_i)\}, \qquad i = 1, \dots M_\gamma.$$

Denote by $\widetilde{\tau}_k^\varepsilon$ the k-th return of $\widetilde{\beta}^\varepsilon$ to the origin. We also set $\sigma_i^\varepsilon := \widetilde{\tau}_{n_i^\varepsilon}^\varepsilon$ and $\sigma_{M_\gamma+1}^\varepsilon := \widetilde{\tau}_{N_\varepsilon}^\varepsilon = \tau^\varepsilon$.

Note that each of the variables $\sigma_{i+1}^\varepsilon - \sigma_i^\varepsilon$ is bounded below in distribution by $\widetilde{\tau}_{[a_\gamma/2\varepsilon]}^\varepsilon$, and recall that for every $c > 0$, $\widetilde{\tau}_{[c/\varepsilon]}^\varepsilon$ converges in distribution to τ_c. Since $\tau_c > 0$ a.s., we may choose $\delta \in (0, \kappa/2)$ so small that, for ε small,

$$P[\sigma_{i+1}^\varepsilon - \sigma_i^\varepsilon > 2\delta \text{ for every } i \in \{1, \dots, M_\gamma\}] > 1 - \frac{\eta}{5}. \tag{4.11}$$

Write E'_ε for the intersection of the set E_ε with the event considered in (4.11). Notice that on E'_ε we can choose a finite sequence $0 = s_0^\varepsilon < s_1^\varepsilon < \cdots < s_{K_\varepsilon}^\varepsilon = \tau^\varepsilon$ in such a way that $\delta \leq s_j^\varepsilon - s_{j-1}^\varepsilon \leq 2\delta < \kappa$, for every $j \in \{1, \dots, K_\varepsilon\}$, and each interval $[s_{j-1}^\varepsilon, s_j^\varepsilon)$ is contained in exactly one interval $[\sigma_{k-1}^\varepsilon, \sigma_k^\varepsilon)$.

We use the sequence (s_i^ε) to get an upper bound on $\theta(\varepsilon, \delta)$ on the event E'_ε. First observe that for $j \in \{1, \dots, K_\varepsilon\}$,

$$\sup_{s,s' \in [s_{j-1}^\varepsilon, s_j^\varepsilon)} d(\widetilde{W}_s^\varepsilon, \widetilde{W}_{s'}^\varepsilon) \leq \sup_{s,s' \in [s_{j-1}^\varepsilon, s_j^\varepsilon)} |\widetilde{\beta}_s^\varepsilon - \widetilde{\beta}_{s'}^\varepsilon| + \sup_{s,s' \in [s_{j-1}^\varepsilon, s_j^\varepsilon)} \sup_{t \geq 0} |\widetilde{W}_s^\varepsilon(t) - \widetilde{W}_{s'}^\varepsilon(t)|.$$

The first term on the right hand side is bounded above by $\rho \leq \eta/5$ by the definition of E_ε (cf (4.10)) and the property $s_j^\varepsilon - s_{j-1}^\varepsilon < \kappa$. To bound the second term, let $s, s' \in [s_{j-1}^\varepsilon, s_j^\varepsilon)$ and consider first the case when

$$m^\varepsilon(s, s') := \inf_{r \in [s,s']} \widetilde{\beta}_r^\varepsilon > 0.$$

Then $\widetilde{W}_s^\varepsilon(t) = \widetilde{W}_{s'}^\varepsilon(t)$ for every $t \in [0, m^\varepsilon(s, s')]$, and thus

$$\begin{aligned}
&\sup_{t \geq 0} |\widetilde{W}_s^\varepsilon(t) - \widetilde{W}_{s'}^\varepsilon(t)| \\
&\quad \leq \sup_{m^\varepsilon(s,s') \leq t \leq \widetilde{\beta}_s^\varepsilon} |\widetilde{W}_s^\varepsilon(t) - \widetilde{W}_s^\varepsilon(m^\varepsilon(s, s'))| + \sup_{m^\varepsilon(s,s') \leq t \leq \widetilde{\beta}_{s'}^\varepsilon} |\widetilde{W}_{s'}^\varepsilon(t) - \widetilde{W}_{s'}^\varepsilon(m^\varepsilon(s, s'))| \\
&\quad \leq \frac{2\eta}{5}
\end{aligned}$$

again by the definition of E_ε (cf (4.9) and (4.10)). The case $m^\varepsilon(s,s')=0$ is analogous, but we now get the additional term $|\widetilde{W}^\varepsilon_s(0)-\widetilde{W}^\varepsilon_{s'}(0)|$. However, by construction, s and s' belong to the same interval $[\sigma^\varepsilon_{k-1},\sigma^\varepsilon_k)$ and thus $\widetilde{W}^\varepsilon_s(0)$ and $\widetilde{W}^\varepsilon_{s'}(0)$ belong to the same $[y_k,z_k)$, which implies that $|\widetilde{W}^\varepsilon_s(0)-\widetilde{W}^\varepsilon_{s'}(0)|\le\gamma=\eta/5$. Finally, for every $j\in\{1,\dots,K_\varepsilon\}$, we get the bound

$$\sup_{s,s'\in[s^\varepsilon_{j-1},s^\varepsilon_j)} d(\widetilde{W}^\varepsilon_s,\widetilde{W}^\varepsilon_{s'})\le\frac{4\eta}{5}<\eta$$

on E'_ε. It follows that, for ε small,

$$P[\theta(\varepsilon,\delta)\ge\eta]\le P[(E'_\varepsilon)^c]\le\frac{3\eta}{5}<\eta.$$

This completes the proof of (4.8) and of the tightness of the sequence $\widetilde{W}^\varepsilon$.

The remaining assertions of Proposition 4.2 are easy. (i) is clear since $\widetilde{\beta}$ must be the weak limit of $\widetilde{\beta}^\varepsilon$. (ii) follows from the analogous property for $\widetilde{W}^\varepsilon$, and a similar argument applies to (iii). □

4.3 Tightness of the reflected historical processes

Recall that the historical process for the ε-reflected system is the process with values in $M_f(\mathcal{W})$ defined by

$$\widetilde{Y}^\varepsilon_t=\int_0^{\tau_\varepsilon} d\widetilde{L}^{\varepsilon,t}_s\,\delta_{\widetilde{W}^\varepsilon_s}.$$

It is easy to verify that $\widetilde{Y}^\varepsilon$ has right-continuous paths with left limits. The following theorem is a slightly more precise version of Theorem 1.1.

Theorem 4.3. *The sequence of the laws $\widetilde{\mathcal{L}}^\varepsilon_Y$ of $\widetilde{Y}^\varepsilon$, $\varepsilon\in\mathcal{E}$, is tight in the space of probability measures on $\mathbf{D}([0,\infty),M_f(\mathcal{W}))$ and any limit law is supported on $\mathbf{C}([0,\infty),M_f(\mathcal{W}))$. Suppose that $\widetilde{\mathcal{L}}_Y$ is the limit of a subsequence of $\widetilde{\mathcal{L}}^\varepsilon_Y$. By passing to a further subsequence of ε's, if necessary, we may assume that the laws $\widetilde{\mathcal{L}}^\varepsilon_W$ of $\widetilde{W}^\varepsilon$ converge to a law $\widetilde{\mathcal{L}}_W$. Then one can construct on some probability space processes $\widetilde{Y}$ and $\widetilde{W}$ with distributions $\widetilde{\mathcal{L}}_Y$ and $\widetilde{\mathcal{L}}_W$, resp., related by*

$$\widetilde{Y}_t=\int_0^{\tilde{\tau}} d\widetilde{L}^t_s\,\delta_{\widetilde{W}_s},$$

where $(\widetilde{L}^t_s,t\ge0,s\ge0)$ denote the local times of the process $\widetilde{\beta}_s:=\zeta_{\widetilde{W}_s}$, and $\widetilde{\tau}=\inf\{s\ge 0:\widetilde{L}^0_s=a\}$.

Proof. By Proposition 4.2, the laws of $\widetilde{W}^\varepsilon$, $\varepsilon\in\mathcal{E}$ are tight. Hence, from any subsequence of $\mathcal{E}$, we can extract a further subsequence $\mathcal{E}_0$ along which $\widetilde{W}^\varepsilon$ converges in distribution.

We can in fact obtain more. For every $\varepsilon > 0$ and $t \geq 0$, denote by Γ^ε_t, $\widetilde{\Gamma}^\varepsilon_t$ the random measures on $\mathbb{R}_+$ defined by

$$\langle \Gamma^\varepsilon_t, \varphi \rangle = \int_0^{\tau\varepsilon} dL^{\varepsilon,t}_s \, \varphi(s) \ , \qquad \langle \widetilde{\Gamma}^\varepsilon_t, \varphi \rangle = \int_0^{\tilde{\tau}\varepsilon} d\widetilde{L}^{\varepsilon,t}_s \, \varphi(s).$$

(We have $\tau^\varepsilon = \widetilde{\tau}^\varepsilon$ but we prefer to keep a different notation here.) Also define Γ_t by:

$$\langle \Gamma_t, \varphi \rangle = \int_0^{\tau} dL^t_s \, \varphi(s).$$

As a consequence of Lemma 2.1, we know that

$$(\beta^\varepsilon_{\cdot \wedge \tau^\varepsilon}, \Gamma^\varepsilon) \xrightarrow[\varepsilon \to 0]{} (\beta_{\cdot \wedge \tau}, \Gamma)$$

uniformly on $[0, \infty)^2$, a.s. If we replace the pair $(\beta^\varepsilon_{\cdot \wedge \tau^\varepsilon}, \Gamma^\varepsilon)$ by $(\widetilde{\beta}^\varepsilon_{\cdot \wedge \tilde{\tau}^\varepsilon}, \widetilde{\Gamma}^\varepsilon)$ this convergence still holds in distribution in $\mathbf{C}(\mathbb{R}_+, \mathbb{R}) \times \mathbf{D}(\mathbb{R}_+, M_f(\mathbb{R}_+))$. From this observation and standard arguments, we have the joint convergence

$$(\widetilde{W}^\varepsilon, \widetilde{\beta}^\varepsilon, \widetilde{\Gamma}^\varepsilon) \xrightarrow[\varepsilon \to 0, \varepsilon \in \mathcal{E}_0]{\text{(d)}} (\widetilde{W}, \widetilde{\beta}, \widetilde{\Gamma}) \tag{4.12}$$

where

$$\langle \widetilde{\Gamma}_t, \varphi \rangle = \int_0^{\tilde{\tau}} d\widetilde{L}^t_s \, \varphi(s),$$

with the notation introduced in the theorem.

By the Skorohod representation theorem, we can replace for every $\varepsilon \in \mathcal{E}_0$ the triplet $(\widetilde{W}^\varepsilon, \widetilde{\beta}^\varepsilon, \widetilde{\Gamma}^\varepsilon)$ by a new triplet having the same distribution, in such a way that the convergence (4.12) now holds almost surely. Without risk of confusion, we keep the same notation for the new triplets. We claim that we have then

$$\widetilde{Y}^\varepsilon_t = \int \widetilde{\Gamma}^\varepsilon_t(ds) \, \delta_{\widetilde{W}^\varepsilon_s} \xrightarrow[\varepsilon \to 0, \varepsilon \in \mathcal{E}_0]{} \int \widetilde{\Gamma}_t(ds) \, \delta_{\widetilde{W}_s} = \widetilde{Y}_t \tag{4.13}$$

uniformly on compact subsets of $\mathbb{R}_+$, a.s. Clearly Theorem 4.3 follows from (4.13) and the fact that the limiting process $\widetilde{Y}$ that appears in (4.13) is continuous. Both (4.13) and the latter fact are immediate consequences of the convergence (4.12) (now assumed to hold a.s.) and the following "elementary" lemma, whose proof is left to the reader.

Lemma 4.4. *Let $(\gamma^n, n \in \mathbb{N})$ be a sequence in $\mathbf{D}(\mathbb{R}_+, M_f(\mathbb{R}_+))$. Assume that γ^n_t converges as $n \to \infty$ to γ_t, uniformly on every compact of $\mathbb{R}_+$, that $t \to \gamma_t$ is continuous and that the measure γ_t is diffuse, for every $t \in \mathbb{R}_+$. Let E be a Polish space and let $(f_n, n \in \mathbb{N})$*

be a sequence in $\mathbf{D}(\mathbb{R}_+, E)$ *that converges to* f *in* $\mathbf{D}(\mathbb{R}_+, E)$. *For every integer* $n \in \mathbb{N}$ *and every* $t \in \mathbb{R}_+$, *let* $\nu_t^n \in M_f(E)$ *be defined by*

$$\nu_t^n = \int \gamma_t^n(ds)\, \delta_{f_n(s)}.$$

Then ν_t^n *converges as* $n \to \infty$, *uniformly on compact subsets of* $\mathbb{R}_+$, *to the measure* ν_t *defined by*

$$\nu_t = \int \gamma_t(ds)\, \delta_{f(s)}.$$

Furthermore, the mapping $t \to \nu_t$ *is continuous.* □

Remark. We do not know whether the limit law of the sequence $\widetilde{\mathcal{L}}_Y^\varepsilon$ in Theorem 4.3 is unique. A positive answer would give the convergence in distribution of the processes $\widetilde{Y}^\varepsilon$. We can also formulate the problem in terms of the reflected snake. Is there a unique (in law) process $\widetilde{W}$ satisfying properties (i) – (iii) of Proposition 4.2 and such that

$$t \longrightarrow \int_0^{\tilde{\tau}} d\widetilde{L}_s^t\, \delta_{\widetilde{W}_s(t)}$$

is a super-Brownian motion started at μ ?

5. Path properties of the reflected historical process

5.1 Preliminaries

Throughout this section, we consider a process $\widetilde{Y}$ which is a weak limit of the processes $\widetilde{Y}^\varepsilon$ as $\varepsilon \to 0$. According to Theorem 4.3, we may and will assume that $\widetilde{Y}$ is constructed together with the reflected Brownian snake $\widetilde{W}$, in such a way that, for every $t \geq 0$,

$$\widetilde{Y}_t = \int_0^{\tilde{\tau}} d\widetilde{L}_s^t\, \delta_{\widetilde{W}_s}$$

where $(\widetilde{L}_s^t, t \geq 0, s \geq 0)$ denote the local times of the process $\widetilde{\beta}_s := \zeta_{\widetilde{W}_s}$, which is (twice) a reflected Brownian motion stopped at time $\widetilde{\tau} = \inf\{s \geq 0 : \widetilde{L}_s^0 = a\}$.

The process

$$X_t = \int_0^{\tilde{\tau}} d\widetilde{L}_s^t\, \delta_{\widetilde{W}_s(t)}$$

is the weak limit of the processes $X^\varepsilon = \widetilde{X}^\varepsilon$ and therefore must be a super-Brownian motion started at μ.

Let us recall the two key properties of the reflected snake $\widetilde{W}$ (cf Proposition 4.2):

• *Monotonicity property*: Almost surely for every $s \leq s'$ we have $\widetilde{W}_s(t) \leq \widetilde{W}_{s'}(t)$ for every $t \in [0, \widetilde{\beta}_s \wedge \widetilde{\beta}_{s'}]$.

• *Snake property*: The set of discontinuities of the mapping $s \to \widetilde{W}_s$ is contained in the zero set of $\widetilde{\beta}$. Furthermore, if $s < s'$ belong to the same connected component of the complement of the zero set, we have

$$\widetilde{W}_s(t) = \widetilde{W}_{s'}(t) \qquad \text{for every } t \in [0, \inf_{r \in [s,s']} \widetilde{\beta}_r]\,.$$

In order to state a useful preliminary result, we introduce some notation. Let us fix $t > 0$, and denote by (a_i^t, b_i^t), $i \in I_t$ the excursion intervals of $\widetilde{\beta}$ above level t (equivalently, these are the connected components of the open set $\{s \geq 0 : \widetilde{\beta}_s > t\}$). Note that the index set I_t may be empty. For each $i \in I_t$, denote by e_i^t the corresponding excursion

$$e_i^t(s) = \widetilde{\beta}_{(a_i^t + s) \wedge b_i^t} - t\,, \qquad s \geq 0.$$

By the snake property of $\widetilde{W}$, we have

$$\widetilde{W}_s(t) = \widetilde{W}_{a_i^t}(t) =: z_i^t\,, \quad \forall s \in [a_i^t, b_i^t].$$

We denote by $n(de)$ the Itô measure of positive Brownian excursions. We normalize the measure $n(de)$ by declaring that the Poisson point process of excursions from 0, i.e., the family of points $(L^0_{a_i^0}, e_i^0)$, has intensity $ds\, n(de)$.

Proposition 5.1. *Conditionally on X_t, the point measure*

$$\sum_{i \in I_t} \delta_{(z_i^t, e_i^t)}$$

is Poisson with intensity $X_t(dz)\, n(de)$. Consequently, for every Borel subset A of $\mathbb{R}$, the process

$$r \to Z_r^{t,A} = \int \widetilde{Y}_{t+r}(dw)\, \mathbf{1}_A(w(t))$$

is a Feller diffusion started at $X_t(A)$.

We recall that the Feller diffusion is a diffusion process Z on $\mathbb{R}_+$ whose transition kernels are characterized by the Laplace transform: $E[\exp(-\lambda Z_t) | Z_0 = z] = \exp(-z\, u_t(\lambda))$ where

$$u_t(\lambda) = \frac{\lambda}{1 + \frac{1}{2}\lambda t}.$$

The total mass process $\langle X_t, 1 \rangle = \widetilde{L}^t_{\widetilde{\tau}}$ is a Feller diffusion started at a.

Proof. We denote by $\tau_r^{(t)}$ the right-continuous inverse of the function $r \to \widetilde{L}^t_r$. Note that $\tau_r^{(t)} < \infty$ iff $r < \widetilde{L}^t_{\tilde{\tau}} = \langle X_t, 1\rangle$. We can rewrite the definition of X_t as

$$\langle X_t, \varphi\rangle = \int_0^{\widetilde{L}^t_{\tilde{\tau}}} dr\, \varphi(\widetilde{W}_{\tilde{\tau}_r^{(t)}}(t)). \tag{5.1}$$

We also set for every $r \geq 0$,

$$A_r^{(t)} = \int_0^r du\, \mathbf{1}_{\{\widetilde{\beta}_u > t\}}$$

and we let $\gamma_r^{(t)}$ be the right-continuous inverse of the function $r \to A_r^{(t)}$. Finally we set $\widetilde{\beta}_r^{(t)} = \widetilde{\beta}_{\gamma_r^{(t)}} - t$, for every $r \in [0, A_{\tilde{\tau}}^{(t)})$.

We then claim that, conditionally on $\{\widetilde{L}^t_{\tilde{\tau}} = x\}$, the process $(\widetilde{\beta}_r^{(t)}, 0 \leq r < A_{\tilde{\tau}}^{(t)})$ is a reflected Brownian motion started at 0 and killed at the first hitting time of x by its local time at level 0, and is independent of the process $(\widetilde{W}_{\tilde{\tau}_r^{(t)}}(t), 0 \leq r < \widetilde{L}^t_{\tilde{\tau}})$. Except for the independence statement, this is a familiar property of linear Brownian motion: See e.g. Section VI.2 of [RY]. To get the independence property, observe that the analogue of the process $\widetilde{\beta}^{(t)}$ for the ε-reflected system codes (in the sense of Section 2) the genealogy of the descendants of particles at time t. On the other hand, if $\tau^{\varepsilon,(t)}$ denotes the right-continuous inverse of $\widetilde{L}^{\varepsilon,t}$, the process $(\widetilde{W}^{\varepsilon}_{\tau_r^{\varepsilon,(t)}}(t), r \geq 0)$ just enumerates in increasing order the positions of the particles alive at t. The required independence is thus clear at the discrete level of the ε-reflected system, and it is preserved under the passage to the limit (4.12).

To complete the proof, write $\ell_i^{(t)}$ for the local time at 0 of $\widetilde{\beta}^{(t)}$ at the beginning, or the end, of excursion e_i^t. Note that $\tau^{(t)}_{\ell_i^{(t)}} = b_i^t$ and thus

$$z_i^{(t)} = \widetilde{W}_{\tau^{(t)}_{\ell_i^{(t)}}}(t). \tag{5.2}$$

The point measure $\sum \delta_{(\ell_i^{(t)}, e_i^t)}$ is the excursion process of the process $\widetilde{\beta}^{(t)}$. Hence, conditionally on $\{\widetilde{L}^t_{\tilde{\tau}} = x\}$, this point measure is Poisson with intensity $1_{[0,x)}(\ell) d\ell\, n(de)$ and is independent of $(\widetilde{W}_{\tilde{\tau}_r^{(t)}}(t), 0 \leq r < \widetilde{L}^t_{\tilde{\tau}})$. The first part of the lemma then follows from this property, (5.2) and (5.1) (which just says that X_t is the image of the measure $1_{[0,\widetilde{L}^{(t)}_{\tilde{\tau}})}(\ell) d\ell$ under the mapping $\ell \to \widetilde{W}_{\tau_\ell^{(t)}}(t)$).

To get the second assertion of the lemma, note that by the definition of $\widetilde{Y}_{t+r}$,

$$Z_r^{t,A} = \sum_{i \in I_t} \mathbf{1}_{\{z_i^{(t)} \in A\}}\, \ell^r(e_i^{(t)}),$$

where $\ell^r(e_i^{(t)})$ denotes the total local time of excursion $e_i^{(t)}$ at level r. By the first part of the proposition, conditionally on X_t, the random measure

$$\sum_{i\in I_t} \mathbf{1}_{\{z_i^{(t)}\in A\}}\,\delta_{e_i^{(t)}}$$

is Poisson with intensity $X_t(A)\,n(de)$. Hence, conditionally on $\{X_t(A)=x\}$, the process $(Z_r^{t,A}, r\geq 0)$ has the same law as $(L^r_{\tau_x}, r\geq 0)$, and the desired result follows from the celebrated Ray-Knight theorem on Brownian local time. $\square$

Remark. We could easily sharpen the statement of Proposition 5.1 by conditioning on $\widetilde{Y}_t$, or even on $(\widetilde{Y}_u, u\leq t)$ rather than on X_t. We will not need these refinements.

5.2 A priori estimates

By [KS] or [R], we know that, almost surely for every $t>0$, the measure X_t has a continuous density $x_t(y)$ with respect to Lebesgue measure on $\mathbb{R}$, and the family $(x_t(y), t>0, y\in\mathbb{R})$ is jointly continuous. Some of our results will be proved under the following additional assumption:

Assumption (H). *The measure μ has a continuous density $x_0(y)$ with respect to Lebesgue measure.*

Under (H), the family $(x_t(y), t\geq 0, y\in\mathbb{R})$ is jointly continuous (see Theorem 8.3.2 in [Da]).

In order to simplify the statements of the results in this subsection we introduce a constant α. All the results hold for $\alpha=0$, assuming (H). Without this assumption, the results hold for any fixed strictly positive α.

For every $t\geq 0$, $r>0$ and $z\in\mathbb{R}$, we set

$$\psi_{t,t+r}(z)=\sup\{\widetilde{W}_s(t+r) : \widetilde{\beta}_s\geq t+r \text{ and } \widetilde{W}_s(t)<z\},$$

with the usual convention $\sup\emptyset=-\infty$. We also consider the symmetric quantity:

$$\widehat{\psi}_{t,t+r}(z)=\inf\{\widetilde{W}_s(t+r) : \widetilde{\beta}_s\geq t+r \text{ and } \widetilde{W}_s(t)>z\},$$

Proposition 5.2. *Let $\eta\in(0,\frac{1}{2})$ and $c>0$. Then, almost surely, one can choose $\delta_0>0$ small enough so that, for every $\delta\in(0,\delta_0)$, $t\geq\alpha$ and $z\in\mathbb{R}$, the condition $x_t(z)\geq c$ implies*

$$\psi_{t,t+\delta}(z)\geq z-\delta^{\frac{1}{2}-\eta}.$$

Proof. For every $t \geq 0$ and $z \in \mathbb{R}$ set

$$\gamma^{t,z} = \inf\{s \geq 0 : \widetilde{\beta}_s \geq t \text{ and } \widetilde{W}_s(t) \geq z\},$$

with the convention $\inf \emptyset = \widetilde{\tau}$. Using the formula for X_t in terms of $\widetilde{W}$, and then the monotonicity property, we get

$$X_t((-\infty, z]) = \int_0^{\tilde{\tau}} d\widetilde{L}^t_s \, \mathbf{1}_{\{\widetilde{W}_s(t)<z\}} = \int_0^{\tilde{\tau}} d\widetilde{L}^t_s \, \mathbf{1}_{\{s<\gamma^{t,z}\}} = \widetilde{L}^t_{\gamma^{t,z}}.$$

On the other hand, if $s < \gamma^{t,z}$ and $\widetilde{\beta}_s \geq t+\delta$, we have $\widetilde{W}_s(t) < z$ and $\widetilde{W}_s(t+\delta) \leq \psi_{t,t+\delta}(z)$. Therefore,

$$X_{t+\delta}((-\infty, \psi_{t,t+\delta}(z)]) = \int_0^{\tilde{\tau}} d\widetilde{L}^{t+\delta}_s \, \mathbf{1}_{\{\widetilde{W}_s(t+\delta)\leq\psi_{t,t+\delta}(z)\}} \geq \widetilde{L}^{t+\delta}_{\gamma^{t,z}}.$$

Thanks to the Hölder continuity of Brownian local time in the time variable, we can choose $\delta_1 > 0$ so small that, for every $\delta \in (0, \delta_1]$, $t \geq 0$ and $z \in \mathbb{R}$,

$$\widetilde{L}^{t+\delta}_{\gamma^{t,z}} \geq \widetilde{L}^t_{\gamma^{t,z}} - \delta^{\frac{1}{2}-\eta}.$$

By combining all these facts we obtain for every $\delta \in (0, \delta_1]$, $t \geq 0$ and $z \in \mathbb{R}$,

$$X_{t+\delta}((-\infty, \psi_{t,t+\delta}(z)]) \geq X_t((-\infty, z]) - \delta^{\frac{1}{2}-\eta}. \tag{5.3}$$

Note that the set $\{(t, y) : x_t(y) > 0\}$ is contained in the graph of X and is thus relatively compact. By uniform continuity, we can choose $\delta_2 > 0$ small enough so that, for every $t \geq \alpha$ and $z \in \mathbb{R}$, the condition $x_t(z) \geq c$ implies that $x_{t+\delta}(y) > \frac{c}{2}$ for all $\delta \in [0, \delta_2]$ and $y \in [z - \delta_2, z + \delta_2]$. In particular, if $0 < r < \delta_2$ and $\delta \in [0, \delta_2]$,

$$X_{t+\delta}((-\infty, z - r]) < X_{t+\delta}((-\infty, z]) - \frac{c}{2}\, r. \tag{5.4}$$

The proof of the following simple estimate for super-Brownian motion is postponed to the appendix.

Lemma 5.3. *Almost surely there exists $\delta_3 > 0$ such that, for every $t \geq \alpha$, $z \in \mathbb{R}$ and $\delta \in (0, \delta_3)$,*

$$|X_{t+\delta}((-\infty, z]) - X_t((-\infty, z])| \leq \delta^{\frac{1}{2}-\eta}. \tag{5.5}$$

To complete the proof of Proposition 5.2, choose $\delta_0 \in (0, \delta_1 \wedge \delta_2 \wedge \delta_3)$ and also such that $\frac{4}{c}\delta_0^{\frac{1}{2}-\eta} < \delta_2$. Then, if $t \geq \alpha$ and $z \in \mathbb{R}$ are such that $x_t(z) \geq c$, (5.3) and (5.5) give for $\delta \in (0, \delta_0)$,

$$X_{t+\delta}((-\infty, \psi_{t,t+\delta}(z)]) \geq X_{t+\delta}((-\infty, z]) - 2\,\delta^{\frac{1}{2}-\eta}.$$

Using (5.4) with $r = \frac{4}{c}\,\delta^{\frac{1}{2}-\eta}$, we get

$$X_{t+\delta}((-\infty, \psi_{t,t+\delta}(z)]) > X_{t+\delta}((-\infty, z - \frac{4}{c}\,\delta^{\frac{1}{2}-\eta}]),$$

which implies

$$\psi_{t,t+\delta}(z) \geq z - \frac{4}{c}\,\delta^{\frac{1}{2}-\eta}.$$

By replacing η with $\eta' \in (0,\eta)$ we can get rid of the factor $\frac{4}{c}$. $\square$

We can immediately use Proposition 5.2 to derive some useful results on continuity properties of the paths $\widetilde{W}_s$. Note that, if $s \in (0,\widetilde{\tau})$ is such that $\widetilde{\beta}_s \geq t+r$ and $\widetilde{W}_s(t) \geq z$, the monotonicity property of the reflected snake implies that $\widetilde{W}_s(t+r) \geq \psi_{t,t+r}(z)$. Using Proposition 5.2 and the symmetric result for $\widehat{\psi}_{t,t+r}(z)$, we get the following corollary. Recall that we take $\alpha = 0$ if (H) is assumed to hold and $\alpha > 0$ otherwise.

Corollary 5.4. *Let $\eta \in (0,\frac{1}{2})$ and $c > 0$. Then almost surely we can choose δ_0 small enough so that, for every $t \geq \alpha$ and every $s \in (0,\widetilde{\tau})$ such that $\widetilde{\beta}_s > t$ and $x_t(\widetilde{W}_s(t)) \geq c$, we have for every $r \in [t,(t+\delta_0)\wedge \widetilde{\beta}_s]$,*

$$|\widetilde{W}_s(r) - \widetilde{W}_s(t)| \leq (r-t)^{\frac{1}{2}-\eta}.$$

5.3 The key technical lemma

Our aim is to refine the a priori estimates that were derived in the previous subsection. To this end, we will need a crucial technical lemma (Lemma 5.7 below), whose proof requires coming back to the approximating branching particle systems. Recall the notation $(W^\varepsilon, \beta^\varepsilon, Y^\varepsilon)$ of the previous sections. A much simplified version of the arguments of Section 4 yields the convergence in distribution

$$(W^\varepsilon, Y^\varepsilon) \xrightarrow[\varepsilon\to 0]{\text{(d)}} (W, Y),$$

where W is a minor modification of the Brownian snake of [L2] (to be precise, W is obtained by concatenating a Poisson point process of Brownian snake excursions with intensity $\int \mu(dy)\,\mathbb{N}_y$, in the notation of [L2]) and Y is the historical super-Brownian motion connected to W via the formula

$$Y_t = \int_0^\tau dL^t_s\,\delta_{W_s},$$

where $(L^t_s, t \geq 0, s \geq 0)$ are the local times of the lifetime process $\beta_s = \zeta_{W_s}$, which is a reflected Brownian motion stopped at time $\tau = \inf\{s \geq 0 : L^0_s = a\}$. (Our notation is

slightly inconsistent with the previous sections, where β was not stopped, but this should cause no confusion.)

On the other hand (cf the proof of Theorem 4.3), we may and will assume that there is a sequence $\mathcal{E}_0$ of values of ε such that

$$(\widetilde{W}^\varepsilon, \widetilde{Y}^\varepsilon) \xrightarrow[\varepsilon\to 0, \varepsilon\in\mathcal{E}_0]{\text{(d)}} (\widetilde{W}, \widetilde{Y}).$$

By a compactness argument, and replacing the sequence $\mathcal{E}_0$ by a subsequence if necessary, we have also

$$(W^\varepsilon, Y^\varepsilon, \widetilde{W}^\varepsilon, \widetilde{Y}^\varepsilon) \xrightarrow[\varepsilon\to 0, \varepsilon\in\mathcal{E}_0]{\text{(d)}} (W, Y, \widetilde{W}, \widetilde{Y}).$$

By the Skorohod representation theorem, we can for every $\varepsilon \in \mathcal{E}_0$ find a 4-tuple which has the same distribution as $(W^\varepsilon, Y^\varepsilon, \widetilde{W}^\varepsilon, \widetilde{Y}^\varepsilon)$ (and for which we keep the same notation), in such a way that the previous convergence now holds a.s.:

$$(W^\varepsilon, Y^\varepsilon, \widetilde{W}^\varepsilon, \widetilde{Y}^\varepsilon) \xrightarrow[\varepsilon\to 0, \varepsilon\in\mathcal{E}_0]{\text{(a.s.)}} (W, Y, \widetilde{W}, \widetilde{Y}). \tag{5.6}$$

From now on we will restrict our attention to values of ε in the sequence $\mathcal{E}_0$ and assume that (5.6) holds. From the equality $\widetilde{X}^\varepsilon = X^\varepsilon$, we also have

$$\int_0^\tau dL_s^t\, \delta_{W_s(t)} = \int_0^{\tilde{\tau}} d\widetilde{L}_s^t\, \delta_{\widetilde{W}_s(t)} = X_t,$$

and we see that τ coincides with $\widetilde{\tau}$.

We introduce the following more restrictive version of Assumption (H):

Assumption (H'). *The measure μ has a continuous density $x_0(y)$, which is Hölder continuous with exponent $\frac{1}{2} - \delta$, for every $\delta > 0$.*

As in the case of Assumption (H), in order to be able to use a single statement for a result with or without Assumption (H'), we take $\alpha = 0$ if (H') holds and otherwise we let α be a fixed strictly positive constant. We also fix a constant $c \in (0,1)$.

Let η, η', ρ be three positive constants, with $0 < \eta < \eta' < 1/4$ and $\rho \in (0, \frac{1}{2})$. For every $\delta \in (0,1)$, we denote by $E(\delta)$ the event on which the following three conditions hold.

A. For every $s \geq 0$, $t \in [0, \beta_s]$, and $r \in [t, (t+\delta)\wedge\beta_s]$,

$$|W_s(r) - W_s(t)| \leq \frac{1}{2}\,(r-t)^{\frac{1}{2}-\eta}.$$

B. For every $t \geq \alpha$ and $s \geq 0$ such that $\widetilde{\beta}_s > t$ and $x_t(\widetilde{W}_s(t)) \geq c$, we have for every $r \in [t, (t+\delta)\wedge\widetilde{\beta}_s]$,

$$|\widetilde{W}_s(r) - \widetilde{W}_s(t)| \leq (r-t)^{\frac{1}{2}-\eta}.$$

C. For every $t \geq \alpha$, $z \in \mathbb{R}$, and $y \in [z - \delta^{\frac{1}{2}-\eta'}, z + \delta^{\frac{1}{2}-\eta'}]$,

$$|x_t(z) - x_t(y)| \leq |z - y|^{\frac{1}{2}-\rho}.$$

Note that the sets $E(\delta)$ are decreasing in δ. We have $P[\bigcup_n E(2^{-n})] = 1$. The fact that properties A and B hold for δ small enough follows from the Hölder continuity properties of the Brownian snake paths (cf (2.7)) and Corollary 5.4 respectively. For property C, see Theorem 8.3.2 in [Da] when $\alpha > 0$. When $\alpha = 0$ (then (H') is in force), the desired Hölder continuity of the densities is easily obtained from formula (8.3.5b) of [Da] by using the techniques of [KS].

Throughout this subsection, we fix $\delta \in (0,1)$, $t \geq \alpha$ and $z \in \mathbb{R}$. We plan to improve the estimates obtained on $\psi_{t,t+\delta}(z)$ in the previous subsection. We set

$$\gamma = \delta^{\frac{1}{2}-\eta'}$$

and we assume that δ has been chosen small enough so that $\gamma > 4\,\delta^{\frac{1}{2}-\eta}$. Then, for every $r \in [t, t+\delta]$, we set

$$X_r^* = \int Y_r(dw)\, \mathbf{1}_{\{w(t) \in (z-\gamma, z+\gamma)\}}\, \delta_{w(r)}.$$

The random measure X_r^* corresponds, for the historical super-Brownian motion Y, to the contribution of those particles alive at time r whose ancestor at time t lies in the interval $(z-\gamma, z+\gamma)$. Note that X_t^* is simply the restriction of X_t to $(z-\gamma, z+\gamma)$.

Our goal is to compare $X_{t+\delta}^*((-\infty, \psi_{t,t+\delta}(z)])$ to $X_t^*((-\infty, z])$ in the same way as we compared $X_{t+\delta}((-\infty, \psi_{t,t+\delta}(z)])$ to $X_t((-\infty, z])$ in (5.3) above. Unfortunately, the argument has to be significantly more complicated.

We set for every $\varepsilon > 0$,

$$\psi_{t,t+\delta}^{\varepsilon}(z) = \sup\{\widetilde{W}_s^{\varepsilon}(t+\delta) : \widetilde{\beta}_s^{\varepsilon} \geq t+\delta \text{ and } \widetilde{W}_s^{\varepsilon}(t) < z\},$$

which represents for the ε-reflected system the right-most position among those particles alive at time $t+\delta$ which are descendants of the particles located to the left of z at time t.

Lemma 5.5. *We have*

$$\psi_{t,t+\delta}(z) = \lim_{\varepsilon \to 0} \psi_{t,t+\delta}^{\varepsilon}(z) \qquad a.s.$$

Proof. This is basically a consequence of the convergence of $\widetilde{W}^{\varepsilon}$ towards $\widetilde{W}$, which entails the convergence of $\widetilde{\beta}^{\varepsilon}$ to $\widetilde{\beta}$. We also use the fact that in the definition of $\psi_{t,t+\delta}(z)$, i.e.,

$$\psi_{t,t+\delta}(z) = \sup\{\widetilde{W}_s(t+\delta) : \widetilde{\beta}_s \geq t+\delta \text{ and } \widetilde{W}_s(t) < z\},$$

we can replace the weak inequality $\widetilde{\beta}_s \geq t+\delta$ by a strict one, and/or the strict inequality $\widetilde{W}_s(t) < z$ by a weak one. To justify this, note that:

(a) Almost surely, every s such that $\widetilde{\beta}_s = t+\delta$ is the limit of a sequence s_n such that $\widetilde{\beta}_{s_n} > t+\delta$ (simply because $t+\delta$ cannot be a local maximum of $\widetilde{\beta}$).

(b) With probability 1, there is no value of s such that $\widetilde{W}_s(t) = z$ and $\widetilde{\beta}_s \geq t+\delta$ (this immediately follows from Proposition 5.1).

We leave details to the reader. □

We now introduce a different approximation of $\psi_{t,t+\delta}(z)$. We consider in the (non-reflected) ε-system those particles which are located at time t in the interval $(z-\gamma, z+\gamma)$, and the descendants of these particles after time t. With this branching particle system (evolving over the time interval $[t,\infty)$), we can associate a reflected system in the way explained in Subsection 3.1. We denote by $\psi^{*,\varepsilon}_{t,t+\delta}(z)$ the position in this new reflected system of the right-most particle at time $t+\delta$, among those particles which are descendants of the particles located to the left of z at time t.

For every $r>0$, we set $\underline{x}(t,z,r) = \inf\{x_t(y) : |y-z| \leq r\}$ and $\overline{x}(t,z,r) = \sup\{x_t(y) : |y-z| \leq r\}$.

Lemma 5.6. *We have*

$$P\Big[\Big(\limsup_{\varepsilon\downarrow 0}\{\psi^{*,\varepsilon}_{t,t+\delta}(z) \neq \psi^{\varepsilon}_{t,t+\delta}(z)\}\Big) \cap E(\delta) \cap \{\underline{x}(t,z,\delta^{1/2}) \geq c\}\Big] \leq 2\exp(-2c\delta^{-1/2}).$$

Proof. We introduce the following events:

$$\Lambda^+ = \{\exists s \geq 0 : \widetilde{\beta}_s > t+\delta \text{ and } \widetilde{W}_s(t) \in (z-\delta^{1/2}, z)\},$$

and

$$\Lambda^- = \{\exists s \geq 0 : \widetilde{\beta}_s > t+\delta \text{ and } \widetilde{W}_s(t) \in (z, z+\delta^{1/2})\}.$$

We first verify that a.s.,

$$\Big(\Big(\limsup_{\varepsilon\downarrow 0}\{\psi^{*,\varepsilon}_{t,t+\delta}(z) \neq \psi^{\varepsilon}_{t,t+\delta}(z)\}\Big) \cap E(\delta) \cap \{\underline{x}(t,z,\delta^{1/2}) \geq c\}\Big) \subset (\Lambda^+ \cap \Lambda^-)^c. \qquad (5.7)$$

Suppose that $\Lambda^+ \cap \Lambda^- \cap E(\delta) \cap \{\underline{x}(t,z,\delta^{1/2}) \geq c\}$ holds. Then, there exists $s_1 \geq 0$ such that $\widetilde{\beta}_{s_1} > t+\delta$ and $W_{s_1}(t) \in (z-\delta^{1/2}, z)$. From property B in the definition of $E(\delta)$ we also have $|\widetilde{W}_{s_1}(r) - z| < 2\delta^{\frac{1}{2}-\eta}$ for every $r \in [t,t+\delta]$. Similarly, there exists $s_2 \geq 0$ such that $\widetilde{\beta}_{s_2} > t+\delta$, $W_{s_2}(t) \in (z, z+\delta^{1/2})$ and $|\widetilde{W}_{s_2}(r) - z| < 2\delta^{\frac{1}{2}-\eta}$ for every $r \in [t,t+\delta]$. By the convergence (5.6), the same properties hold for $\varepsilon > 0$ small enough, if we replace $\widetilde{W}_{s_i}$ and $\widetilde{\beta}_{s_i}$ by $\widetilde{W}^{\varepsilon}_{s_i}$ and $\widetilde{\beta}^{\varepsilon}_{s_i}$ respectively.

On the other hand, by property A of the definition of $E(\delta)$ and the convergence (5.6), we have also for ε small enough, for every s such that $\beta^\varepsilon_s \geq t$ and every $r \in [t, (t+\delta) \wedge \widetilde{\beta}^\varepsilon_s]$,

$$|W^\varepsilon_s(r) - W^\varepsilon_s(t)| \leq \delta^{\frac{1}{2}-\eta}.$$

In particular, if s is such that $\beta^\varepsilon_s \geq t$ and $|W^\varepsilon_s(t) - z| \geq \gamma \geq 4\,\delta^{\frac{1}{2}-\eta}$, we have for every $r \in [t, (t+\delta) \wedge \widetilde{\beta}^\varepsilon_s]$,

$$|W^\varepsilon_s(r) - z| > 2\,\delta^{\frac{1}{2}-\eta}.$$

We have shown that, on the event $\Lambda^+ \cap \Lambda^- \cap E(\delta) \cap \{\underline{x}(t, z, \delta^{1/2}) \geq c\}$, provided that ε is small enough:

- There exist s_1 and s_2 such that $\widetilde{\beta}_{s_1} > t+\delta$, $\widetilde{\beta}_{s_2} > t+\delta$ and

$$\begin{aligned} &W^\varepsilon_{s_1}(t) \in (z - \delta^{1/2}, z)\ , \qquad |\widetilde{W}^\varepsilon_{s_1}(r) - z| < 2\delta^{\frac{1}{2}-\eta},\ \forall r \in [t, t+\delta] \\ &W^\varepsilon_{s_2}(t) \in (z, z + \delta^{1/2})\ , \qquad |\widetilde{W}^\varepsilon_{s_2}(r) - z| < 2\delta^{\frac{1}{2}-\eta},\ \forall r \in [t, t+\delta]. \end{aligned}$$

- For every $s \geq 0$ such that $\beta^\varepsilon_s \geq t$ and $|W^\varepsilon_s(t) - z| \geq \gamma$,

$$|W^\varepsilon_s(r) - z| > 2\,\delta^{\frac{1}{2}-\eta},\ \forall r \in [t, (t+\delta) \wedge \widetilde{\beta}^\varepsilon_s].$$

These properties allow us to apply Lemma 3.1. In the context of that lemma, the original system is the ε-system considered after time t, the new (restricted) system consists of the descendants of the particles which are located at time t in the interval $(z-\gamma, z+\gamma)$, and we take $I = (z - 2\delta^{\frac{1}{2}-\eta}, z + 2\delta^{\frac{1}{2}-\eta})$. Lemma 3.1 and the previous properties imply that the restrictions of the paths $\widetilde{W}^\varepsilon_{s_1}$ and $\widetilde{W}^\varepsilon_{s_2}$ to $[t, t+\delta]$ still appear as restrictions of reflected historical paths in the new system. Note that in the definition of $\psi^\varepsilon_{t,t+\delta}(z)$, respectively of $\psi^{*,\varepsilon}_{t,t+\delta}(z)$, we may restrict our attention to those reflected historical paths between times t and $t+\delta$ in the original system, resp. in the new system, whose value at time t lies in the interval $[\widetilde{W}^\varepsilon_{s_1}(t), z)$ (this is so because of the monotonicity property of reflected historical paths). Any such path is bounded below and above by $\widetilde{W}^\varepsilon_{s_1}$ and $\widetilde{W}^\varepsilon_{s_2}$ respectively, on the time interval $[t, t+\delta]$. By Lemma 3.1 again, the class of paths that we need to consider is exactly the same for both the original system and the new one. This is enough to conclude that $\psi^{*,\varepsilon}_{t,t+\delta}(z) = \psi^\varepsilon_{t,t+\delta}(z)$, and we get our claim (5.7).

It follows from (5.7) that the probability considered in the lemma is bounded above by

$$P[(\Lambda^+ \cap \Lambda^-)^c \cap \{\underline{x}(t, z, \delta^{1/2}) \geq c\}].$$

By the construction of $\widetilde{Y}$, we have

$$\int_0^{\tilde{\tau}} d\widetilde{L}^{t+\delta}_s\, \mathbf{1}_{\{\widetilde{W}_s(t) \in (z-\delta^{1/2}, z)\}} = \int \widetilde{Y}_{t+\delta}(dw)\, \mathbf{1}_{\{w(t) \in (z-\delta^{1/2}, z)\}}.$$

Hence the event Λ^+ certainly holds if

$$\int \widetilde{Y}_{t+\delta}(dw)\,\mathbf{1}_{\{w(t)\in(z-\delta^{1/2},z)\}} > 0.$$

It follows that

$$P[(\Lambda^+)^c \cap \{\underline{x}(t,z,\delta^{1/2}) \geq c\}] \leq P[\{\int \widetilde{Y}_{t+\delta}(dw)\,\mathbf{1}_{\{w(t)\in(z-\delta^{1/2},z)\}} = 0\} \cap \{\underline{x}(t,z,\delta^{1/2}) \geq c\}],$$

and a similar bound holds if we replace Λ^+ by Λ^-. By Proposition 5.1, the last quantity is bounded above by the probability that a Feller diffusion started at $c\,\delta^{1/2}$ vanishes at time δ. This probability is equal to $\exp(-2c\delta^{-1/2})$, which completes the proof. $\square$

We can now state the key lemma. We fix still another constant $\eta'' \in (\eta', 1/4)$.

Lemma 5.7. *There exist two positive constants C and κ, that depend only on c, η, η', η'' and ρ, such that*

$$\begin{aligned} P\big[\{|X^*_{t+\delta}((-\infty,\psi_{t,t+\delta}(z)]) - X^*_t((-\infty,z])| > \delta^{\frac{3}{4}-\eta''}\} \\ \cap E(\delta) \cap \{\underline{x}(t,z,\delta^{1/2}) \geq c\} \cap \{\overline{x}(t,z,\gamma) \leq c^{-1}\}\big] \leq C\,\exp(-\delta^{-\kappa}). \end{aligned}$$

Proof. For every $r \in [t, t+\delta]$, set

$$X^{*,\varepsilon}_r = \int Y^\varepsilon_r(dw)\,\mathbf{1}_{\{w(t)\in(z-\gamma,z+\gamma)\}}\delta_{w(r)},$$

which represents the contribution at time r of the descendants (in the non-reflected system) of particles which are located in $(z-\gamma, z+\gamma)$ at time t. From the convergence of Y^ε to Y, and the fact that $\int Y_r(dw)\,\mathbf{1}_{\{w(t)=z\pm\gamma\}} = 0$, one can easily show that for every $r \in [t, t+\delta]$, the measures $X^{*,\varepsilon}_r$ converge weakly to X^*_r. In particular, a.s. for every $y \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0} X^{*,\varepsilon}_{t+\delta}((-\infty,y]) = X^*_{t+\delta}((-\infty,y]).$$

From Lemma 5.5 and Lemma 5.6, we get that on the set $E(\delta) \cap \{\underline{x}(t,z,\delta^{1/2} \geq c\}$, we have the convergence

$$\lim_{\varepsilon\to 0} X^{*,\varepsilon}_{t+\delta}((-\infty,\psi^{*,\varepsilon}_{t,t+\delta}(z)]) = X^*_{t+\delta}((-\infty,\psi_{t,t+\delta}(z)]),$$

except possibly on a set of measure at most $2\,\exp(-2c\delta^{-1/2})$.

However, by the definition of $\psi^{*,\varepsilon}_{t,t+\delta}(z)$, and the monotonicity property of reflected systems, the quantity $X^{*,\varepsilon}_{t+\delta}((-\infty,\psi^{*,\varepsilon}_{t,t+\delta}(z)])$ is equal to ε times the number of descendants at time $t+\delta$ of the particles present at time t in $(z-\gamma, z)$, for the ε-reflected system constructed

over the time interval $[t,\infty)$ from the particles present at time t in $(z-\gamma, z+\gamma)$. Since the law of the branching evolution is the same for the reflected system as for the original one, we see that conditionally on $\{X^{*,\varepsilon}_t((-\infty,z]) = \varepsilon\, k\}$, the variable $X^{*,\varepsilon}_{t+\delta}((-\infty,\psi^{*,\varepsilon}_{t,t+\delta}(z)])$ is distributed as $\varepsilon\, Z^{(\varepsilon,k)}_\delta$, where $Z^{(\varepsilon,k)}$ denotes a Galton-Watson process with critical binary branching at rate ε^{-1} and initial value k. Recall that $X^{*,\varepsilon}_t((-\infty,z])$ converges a.s. to $X^*_t((-\infty,z])$. By standard limit theorems for Galton-Watson processes,

$$(X^{*,\varepsilon}_t((-\infty,z]), X^{*,\varepsilon}_{t+\delta}((-\infty,\psi^{*,\varepsilon}_{t,t+\delta}(z)])) \xrightarrow[\varepsilon\to 0]{\rm (d)} (X^*_t((-\infty,z]), U),$$

where conditionally on $X^*_t((-\infty,z]) = u$, the variable U is distributed as the value at time δ of a Feller diffusion started at u.

Note that $X^*_t((-\infty,z]) = X_t((z-\gamma,z])$ and that on the set $\{\overline{x}(t,z,\gamma) \leq c^{-1}\}$ we have $X^*_t((-\infty,z]) \leq c^{-1}\gamma = c^{-1}\delta^{\frac12-\eta'}$. Elementary estimates on the Feller diffusion, using only the form of the Laplace transform of the semigroup (see the appendix for very similar estimates) show that

$$P\big[\{X^*_t((-\infty,z]) \leq c^{-1}\delta^{\frac12-\eta'}\}\cap\{|U - X^*_t((-\infty,z])| \geq \delta^{\frac34-\eta''}\}\big] \leq C'\,\exp(-\delta^{-\kappa'}),$$

where the constants C' and $\kappa'>0$ depend only on c,η' and η''.

To complete the proof of the lemma, we write

$$\begin{aligned}
&P\big[\{|X^*_{t+\delta}((-\infty,\psi_{t,t+\delta}(z)]) - X^*_t((-\infty,z])| > \delta^{\frac34-\eta''}\}\\
&\qquad\qquad\qquad\qquad \cap E(\delta)\cap\{\underline{x}(t,z,\delta^{1/2})\geq c\}\cap\{\overline{x}(t,z,\gamma)\leq c^{-1}\}\big]\\
&\leq 2\,\exp(-2c\delta^{-1/2}) + P\big[\{\liminf_{\varepsilon\to0}|X^{*,\varepsilon}_{t+\delta}((-\infty,\psi^{*,\varepsilon}_{t,t+\delta}(z)]) - X^*_t((-\infty,z])| > \delta^{\frac34-\eta''}\}\\
&\qquad\qquad\qquad\qquad \cap\{X^*_t((-\infty,z])\leq c^{-1}\delta^{\frac12-\eta'}\}\big]\\
&\leq 2\,\exp(-2c\delta^{-1/2}) + \liminf_{\varepsilon\to0} P\big[\{|X^{*,\varepsilon}_{t+\delta}((-\infty,\psi^{*,\varepsilon}_{t,t+\delta}(z)]) - X^*_t((-\infty,z])| > \delta^{\frac34-\eta''}\}\\
&\qquad\qquad\qquad\qquad \cap\{X^*_t((-\infty,z])\leq c^{-1}\delta^{\frac12-\eta'}\}\big]\\
&\leq 2\,\exp(-2c\delta^{-1/2}) + P\big[\{X^*_t((-\infty,z])\leq c^{-1}\delta^{\frac12-\eta'}\}\cap\{|U - X^*_t((-\infty,z])|\geq\delta^{\frac34-\eta''}\}\big]\\
&\leq 2\,\exp(-2c\delta^{-1/2}) + C'\,\exp(-\delta^{-\kappa'}).
\end{aligned}$$

□

5.4 The main result

We keep the notation introduced in the previous subsection. The reals $t\geq\alpha$, $\delta\in(0,1)$ and $z\in\mathbb{R}$ are fixed for the moment.

Lemma 5.8. *Assume that $\eta'' > \frac32\eta' + \frac12\rho$. There exist two constants $\overline{C}$ and $\overline{\kappa}>0$, that depend only on c,η',η'' and ρ, such that*

$$P\big[\{X^*_{t+\delta}((-\infty,z]) \geq X^*_t((-\infty,z]) + \delta^{\frac34-\eta''}\}\cap E(\delta)\cap\{\overline{x}(t,z,\gamma)\leq c^{-1}\}\big] \leq \overline{C}\exp(-\delta^{-\overline{\kappa}}).$$

The proof of this lemma is an application of standard techniques in the theory of super-Brownian motion. See the appendix for a detailed argument.

Proposition 5.9. *Under the assumptions of Lemma 5.8, there exist two constants C_0 and $\kappa_0 > 0$, that depend only on c, η, η', η'' and ρ, such that*

$$P\big[\{\psi_{t,t+\delta}(z) < z - \frac{2}{c}\delta^{\frac{3}{4}-\eta''}\} \cap E(\delta) \\ \cap \{c \leq \underline{x}(t,z,\gamma) \leq \overline{x}(t,z,\gamma) \leq c^{-1}\} \cap \{\underline{x}(t+\delta,z,\gamma) > c\}\big] \leq C_0 \exp(-\delta^{-\kappa_0}).$$

Proof. Our argument is very similar to the proof of Proposition 5.2. We will assume that the event $E(\delta) \cap \{c \leq \underline{x}(t,z,\gamma) \leq \overline{x}(t,z,\gamma) \leq c^{-1}\}$ holds. By Lemmas 5.7 and 5.8, we have on this set

$$X^*_{t+\delta}((-\infty,z]) \leq X^*_t((-\infty,z]) + \delta^{\frac{3}{4}-\eta''} \leq X^*_{t+\delta}((-\infty,\psi_{t,t+\delta}(z)]) + 2\,\delta^{\frac{3}{4}-\eta''} \qquad (5.8)$$

except possibly on a set of probability at most $C\,\exp(-\delta^{-\kappa}) + \overline{C}\,\exp(-\delta^{-\overline{\kappa}})$.

On the other hand, condition A in the definition of $E(\delta)$ (and the fact that $\gamma > 4\,\delta^{\frac{1}{2}-\eta}$) ensures that the measures $X^*_{t+\delta}$ and $X_{t+\delta}$ coincide over the interval $(z - \frac{\gamma}{2}, z + \frac{\gamma}{2})$. Hence, on the event $\{\underline{x}(t+\delta,z,\gamma) > c\}$, we get

$$X^*_{t+\delta}((-\infty,z]) - 2\,\delta^{\frac{3}{4}-\eta''} > X^*_{t+\delta}((-\infty, z - \frac{2}{c}\delta^{\frac{3}{4}-\eta''}]),$$

provided that δ is small enough so that $\frac{2}{c}\delta^{\frac{3}{4}-\eta''} \leq \frac{\gamma}{2}$. On the set where (5.8) holds, we get

$$X^*_{t+\delta}((-\infty,\psi_{t,t+\delta}(z)]) > X^*_{t+\delta}((-\infty, z - \frac{2}{c}\delta^{\frac{3}{4}-\eta''}]),$$

and the desired result follows. □

We now come to the main result of this section, which is a refinement of Corollary 5.4. Recall our conventions concerning α—this constant is equal 0 if (H') is assumed to hold and otherwise α is a fixed strictly positive constant.

Theorem 5.10. *Let $\lambda > 0$ and $c \in (0,1)$. Then a.s. we can choose δ_0 small enough so that, for every $t \geq \alpha$ and every $s \in (0,\tau)$ such that $\widetilde{\beta}_s > t$ and $x_t(\widetilde{W}_s(t)) \geq c$, we have for every $r \in [t,(t+\delta_0) \wedge \widetilde{\beta}_s]$,*

$$|\widetilde{W}_s(r) - \widetilde{W}_s(t)| \leq (r-t)^{\frac{3}{4}-\lambda}.$$

Proof. We can choose η,η',η'' with $0<\eta<\eta'<\eta''<\lambda$ and $\rho\in(0,\frac{1}{2})$ such that the assumptions of Lemma 5.8 hold. We then apply the estimate of Proposition 5.9 with $\delta=2^{-n}$ (n large enough) to all reals $t\in[\alpha,n]$, $z\in[-n,n]$ of the form $t=k2^{-n}$, $z=j2^{-n}$. We have already observed that $P[\bigcup_n E(2^{-n})]=1$. Furthermore, if we assume that $c\leq x_t(z)\leq c^{-1}$ we will have $\underline{x}(t,z,2^{-n(\frac{1}{2}-\eta')})\geq c/2$, $\overline{x}(t,z,2^{-n(\frac{1}{2}-\eta')})\leq 2/c$, and $\underline{x}(t+2^{-n},z,2^{-n(\frac{1}{2}-\eta')})\geq c/2$, for all n sufficiently large (depending on ω but not on t and z). Then, by combining the estimate of Proposition 5.9 with the Borel-Cantelli lemma, we obtain the following property: There exists an integer $n_0(\omega)$ such that for every $n\geq n_0(\omega)$, for every $t=k2^{-n}$, $z=j2^{-n}$ with $t\in[\alpha,n]$, $z\in[-n,n]$, the condition $c\leq x_t(z)\leq c^{-1}$ implies

$$\psi_{t,t+2^{-n}}(z)\geq z-(2^{-n})^{\frac{3}{4}-\lambda}.$$

Since the densities $x_r(y)$ are bounded over $[\alpha,\infty)\times\mathbb{R}$, a simple argument shows that we can drop the condition $x_t(z)\leq c^{-1}$ in the previous assertion.

Then, if $s\geq 0$ is such that $\widetilde{\beta}_s\geq t+2^{-n}$, where t is of the form $t=k2^{-n}$, we let $z=j2^{-n}$ be such that $z<\widetilde{W}_s(t)\leq z+2^{-n}$. If n is large enough (again independently of the choice of s and t), the condition $x_t(\widetilde{W}_s(t))\geq 2c$ will imply $x_t(z)>c$. Then, by the definition of $\psi_{t,t+\delta}(z)$ and the preceding estimate,

$$\widetilde{W}_s(t+2^{-n})\geq\psi_{t,t+2^{-n}}(z)\geq\widetilde{W}_s(t)-2^{-n}-(2^{-n})^{\frac{3}{4}-\lambda}.$$

Thanks to this observation and a symmetry argument, we obtain that a.s. for n large enough, for every $t\geq\alpha$ of the form $t=k2^{-n}$ and every $s\geq 0$ such that $\widetilde{\beta}_s\geq t+2^{-n}$ and $x_t(\widetilde{W}_s(t))\geq 2c$,

$$|\widetilde{W}_s(t+2^{-n})-\widetilde{W}_s(t)|\leq 2\,(2^{-n})^{\frac{3}{4}-\lambda}.$$

The statement of Theorem 5.10 now follows easily thanks to the usual chaining argument. $\square$

Theorem 1.2 is an immediate consequence of Theorem 5.10. Note that, by the representation formula for $\widetilde{Y}$ in terms of $\widetilde{W}$, the set $\operatorname{supp}\widetilde{Y}_t$ is contained in $\{\widetilde{W}_s;\widetilde{\beta}_s=t\}$, for every $t>0$, a.s. The comments following the statement of Theorem 1.2 are justified by Proposition 5.1.

6. Branching points

In this last section, we prove Theorem 1.3. As in Section 5, we assume that the process $\widetilde{Y}$ is constructed together with the reflected Brownian snake $\widetilde{W}$, in such a way that we have the representation formula

$$\widetilde{Y}_t=\int_0^{\tilde{\tau}}d\widetilde{L}^t_s\,\delta_{\widetilde{W}_s}.$$

We need a preliminary lemma. If $s_1 < s_2$, we set $m(s_1, s_2) = \inf_{s \in [s_1,s_2]} \widetilde{\beta}_s$.

Lemma 6.1. *Almost surely, for any $t > 0$ and any $s_1 < s_2$ such that $\widetilde{\beta}_{s_1} = \widetilde{\beta}_{s_2} = t$ and $0 < m(s_1, s_2) < t$, we have*

$$x_{m(s_1,s_2)}(\widetilde{W}_{s_1}(m(s_1, s_2))) > 0.$$

Proof. Let $\alpha > 0$ and let $A \geq 1$ be an integer. Write E_A for the event $E_A = \{\mathcal{G} \subset [0, A] \times [-A, A]\}$, where $\mathcal{G}$ is as above the graph of X. It is enough to prove that a.s. on E_A, the following holds:

(P) For any $t > \alpha$ and $s_1 < s_2$ such that $\widetilde{\beta}_{s_1} = \widetilde{\beta}_{s_2} = t$ and $0 < m(s_1, s_2) < t - \alpha$, we have $x_{m(s_1,s_2)}(\widetilde{W}_{s_1}(m(s_1, s_2))) > 0$.

We first introduce some notation. Let e be an excursion, that is a continuous function $e : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $e(s) > 0$ iff $0 < s < \sigma(e)$, for some $\sigma(e) > 0$. Set

$$T_\alpha(e) = \inf\{s \geq 0 : e(s) = \alpha\}$$

and, if $T_\alpha(e) < \infty$,

$$L_\alpha(e) = \sup\{t \geq 0 : e(t) = \alpha\},$$
$$M_\alpha(e) = \inf_{T_\alpha(e) \leq s \leq L_\alpha(e)} e(s)$$

By convention we take $M_\alpha(e) = 0$ if $T_\alpha(e) = \infty$.

Let $r > 0$. Recall the notation I_r and e_i^r, z_i^r, $i \in I_r$ introduced before Proposition 5.1, and for every $c > 0$ and $\delta \in (0, \alpha)$, set

$$N_r^\delta(\alpha, c) = \sum_{i \in I_r} \mathbf{1}_{\{x_r(z_i^r) \leq c\}} \mathbf{1}_{\{0 < M_\alpha(e_i^r) \leq \delta\}}.$$

Proposition 5.1 allows us to conclude that,

$$\begin{aligned} E[N_r^\delta(\alpha, c)\, \mathbf{1}_{E_A}] &\leq E\Big[\sum_{i \in I_r} \mathbf{1}_{\{|z_i^r| \leq A\}}\, \mathbf{1}_{\{x_r(z_i^r) \leq c\}}\, \mathbf{1}_{\{0 < M_\alpha(e_i^r) \leq \delta\}}\Big] \\ &= E\Big[\int_{-A}^{A} dz\, x_t(z)\, \mathbf{1}_{\{x_r(z) \leq c\}}\, n(0 < M_\alpha(e) \leq \delta)\Big] \\ &\leq 2cA\, \delta \alpha^{-2}, \end{aligned}$$

using the easy formula $n(0 < M_\alpha(e) \leq \delta) = \delta \alpha^{-2}$. We apply this estimate with $\delta = 1/k$ (k large enough) and $r = j/k$ for all $j = 1, 2, \ldots, Ak$. It follows that

$$E\Big[\mathbf{1}_{E_A} \sum_{j=1}^{\infty} N_{j/k}^{1/k}(\alpha, c)\Big] \leq 2c\, A^2\, \alpha^{-2}.$$

In particular, if $E_k(\alpha, c, A)$ denotes the event $\{\exists j \geq 1 : N^{1/k}_{j/k}(\alpha, c) \geq 1\} \cap E_A$, we have

$$P\Big[\liminf_{k\to\infty} E_k(\alpha, c, A)\Big] \leq 2c\, A^2\, \alpha^{-2}. \tag{6.1}$$

Suppose that property (P) fails. Then, we may find $t > \alpha$ and $s_1 < s_2$ such that $\widetilde{\beta}_{s_1} = \widetilde{\beta}_{s_2} = t$ and $0 < m(s_1, s_2) < t - \alpha$, and furthermore $x_{m(s_1,s_2)}(\widetilde{W}_{s_1}(m(s_1, s_2))) = 0$. We take j such that $j/k < m(s_1, s_2) \leq (j + 1)/k$, and observe that $x_{j/k}(\widetilde{W}_{s_1}(j/k)) < c$ for all k sufficiently large, by the joint continuity of densities. Hence by considering the excursion of $\widetilde{\beta}$ above level j/k that contains s_1, we see that $N^{1/k}_{j/k}(\alpha, c) \geq 1$ for all k large. Therefore, if $F(\alpha, A)$ denotes the event on which (P) fails, we have

$$P[F(\alpha, A) \cap E_A] \leq P\Big[\liminf_{k\to\infty} E_k(\alpha, c, A))\Big] \leq 2c\, A^2\, \alpha^{-2}.$$

Since c was arbitrary, we have $P[F(\alpha, A) \cap E_A] = 0$, which completes the proof. $\square$

Proof of Theorem 1.3. The representation formula for $\widetilde{Y}_t$ implies that

$$\operatorname{supp} \widetilde{Y}_t = \{\widetilde{W}_s : \widetilde{\beta}_s = t\}.$$

(Note that the set on the right hand side is closed, by the continuity properties of $\widetilde{W}$.) Hence if w_1 and w_2 belong to $\operatorname{supp} \widetilde{Y}_t$ and $w_1 \neq w_2$, we can find s_1 and s_2 such that $\widetilde{\beta}_{s_1} = \widetilde{\beta}_{s_2} = t$, and $\widetilde{W}_{s_1} = w_1$, $\widetilde{W}_{s_2} = w_2$. With no loss of generality, we can assume $s_1 < s_2$. We claim that

$$m(s_1, s_2) = \inf\{r \in [0, t] : w_1(r) \neq w_2(r)\}. \tag{6.2}$$

The inequality $m(s_1, s_2) \leq \inf\{r \in [0, t] : w_1(r) \neq w_2(r)\}$ is immediate from the snake property (when $m(s_1, s_2) = 0$ there is nothing to prove). On the other hand, if we assume that there is a rational $r \in (m(s_1, s_2), t)$ such that $\widetilde{W}_{s_1}(r) = \widetilde{W}_{s_2}(r)$, then the monotonicity property implies $\widetilde{W}_s(r) = \widetilde{W}_{s_1}(r)$ for every $s \in [s_1, s_2]$ such that $\widetilde{\beta}_s \geq r$. Hence,

$$X_r = \int_0^{\tilde{\tau}} d\widetilde{L}^r_s\, \delta_{\widetilde{W}_s(r)} \geq \int_{s_1}^{s_2} d\widetilde{L}^r_s\, \delta_{\widetilde{W}_s(r)} = (\widetilde{L}^r_{s_2} - \widetilde{L}^r_{s_1})\, \delta_{W_{s_1}(r)},$$

which gives a contradiction since $\widetilde{L}^r_{s_2} - \widetilde{L}^r_{s_1} > 0$ by standard properties of linear Brownian motion.

From now on, we assume $m(s_1, s_2) > 0$. Note that we have also $m(s_1, s_2) < t$ since we assumed that $w_1 \neq w_2$. By Lemma 6.1, we have $x_{m(s_1,s_2)}(\widetilde{W}_{s_1}(m(s_1, s_2))) > 0$. By

monotonicity (and the fact that the measure X_r gives no mass to singletons), we get for every $r \in (m(s_1, s_2), t)$,

$$\int_{s_1}^{s_2} d\widetilde{L}_s^r = X_r((\widetilde{W}_{s_1}(r), \widetilde{W}_{s_2}(r))) = \int_{\widetilde{W}_{s_1}(r)}^{\widetilde{W}_{s_2}(r)} dz\, x_r(z),$$

and by the continuity of densities, it follows that

$$\lim_{r \downarrow m(s_1,s_2)} \frac{\widetilde{W}_{s_2}(r) - \widetilde{W}_{s_1}(r)}{\widetilde{L}_{s_2}^r - \widetilde{L}_{s_1}^r} = x_{m(s_1,s_2)}(\widetilde{W}_{s_1}(m(s_1, s_2))) > 0. \tag{6.3}$$

Thanks to (6.3), the behavior of $w_1(r) - w_2(r)$ as $r \downarrow \gamma_{w_1,w_2} = m(s_1, s_2)$ is reduced to that of $\widetilde{L}_{s_2}^r - \widetilde{L}_{s_1}^r$. Write s_0 for the (unique) time in (s_1, s_2) such that $\widetilde{\beta}_{s_0} = m(s_1, s_2)$. Standard results on Brownian path decompositions show that, for events that depend only on the asymptotic σ-field at time 0, the processes $\{\widetilde{\beta}_{s_0-u} - \widetilde{\beta}_{s_0}, u \in [0, s_0 - s_1]\}$ and $\{\widetilde{\beta}_{s_0+u} - \widetilde{\beta}_{s_0}, u \in [0, s_2 - s_0]\}$ behave as two independent 3-dimensional Bessel processes. It follows from this and the Ray-Knight theorem that the process $\delta \to \widetilde{L}_{s_2}^{m(s_1,s_2)+\delta} - \widetilde{L}_{s_1}^{m(s_1,s_2)+\delta}$ has the same local path properties (for δ close to 0) as the sum of two independent squares of 2-dimensional Bessel processes, which is the square of a 4-dimensional Bessel process. If $\delta \to R_\delta$ is the square of a 4-dimensional Bessel process, the law of the iterated logarithm shows that

$$\limsup_{\delta \downarrow 0} \frac{R_\delta}{2\delta \log|\log \delta|} = 1.$$

On the other hand, from the well-known rate of escape for Brownian motion in space (Theorem 6 in [DE] combined with time-inversion), we have for $\alpha > 0$,

$$\lim_{\delta \downarrow 0} \frac{R_\delta}{\delta |\log \delta|^{-1-\alpha}} = \infty.$$

We have just argued that the same properties hold if we replace R_δ with $\widetilde{L}_{s_2}^{m(s_1,s_2)+\delta} - \widetilde{L}_{s_1}^{m(s_1,s_2)+\delta}$. This and (6.3) imply Theorem 1.3. $\square$

Appendix

Proof of Lemma 5.3. For a fixed value of z, the estimate of Lemma 5.3 follows from [P2]. As we need uniformity in z, we will provide a detailed argument. Recall the notation from Subsection 5.2, and especially the conventions concerning the constant α. Recall that $\mathcal{G}$ denotes the graph of X and for every integer $A \geq 1$ consider the event

$$E_A = \{\mathcal{G} \subset [0, A] \times [-A, A];\ \sup_{t \geq \alpha, y \in \mathbb{R}} x_t(y) \leq A\}.$$

Note that $P[E_A] \uparrow 1$ as $A \uparrow \infty$. (We use assumption (H) when $\alpha = 0$.) The key step of the proof is to show the following inequality for all $t \geq \alpha$ and $z \in \mathbb{R}$,

$$P[\{|X_{t+\delta}((-\infty, z]) - X_t((-\infty, z])| \geq \delta^{\frac{1}{2}-\eta}\} \cap E_A] \leq C \exp(-\delta^{-\kappa}) \qquad (A1),$$

where the constants C and $\kappa > 0$ may depend on A but not on t, z and δ. To prove (A1), we may apply the Markov property at time t and reduce the problem to the case $t = 0$. More precisely it is enough to consider a super-Brownian motion $\Gamma = (\Gamma_t, t \geq 0)$ with initial value $\Gamma_0(dz) = g(z)dz$, with a function g bounded above by A and such that $\int g(z)dz \leq 2A^2$, and to prove that for every $\delta \in (0, 1)$,

$$P[|\Gamma_\delta((-\infty, 0]) - \Gamma_0((-\infty, 0])| \geq \delta^{\frac{1}{2}-\eta}] \leq C \exp(-\delta^{-\kappa}). \qquad (A2)$$

Let us first bound $P[\Gamma_\delta((-\infty, 0]) \leq \Gamma_0((-\infty, 0]) - \delta^{\frac{1}{2}-\eta}]$. We know that for every $\lambda > 0$,

$$E[\exp(-\lambda \Gamma_\delta((-\infty, 0]))] = \exp(-\langle \Gamma_0, u_\delta \rangle),$$

where $u_t(z)$ solves the integral equation

$$u_t(z) + \frac{1}{2} E_z\Big[\int_0^t u_{t-r}(B_r)^2\, dr\Big] = \lambda\, P_z[B_t \leq 0],$$

if B is a linear Brownian motion started at z under P_z. The integral equation gives the bound

$$u_t(z) \geq \lambda\, P_z[B_t \leq 0] - \frac{\lambda^2}{2} t.$$

We use this bound in the following estimates,

$$\begin{aligned}
&P[\Gamma_\delta((-\infty, 0]) \leq \Gamma_0((-\infty, 0]) - \delta^{\frac{1}{2}-\eta}] \\
&\leq \exp(-\lambda\, \delta^{\frac{1}{2}-\eta} + \lambda\, \Gamma_0((-\infty, 0]))\, E[\exp(-\lambda \Gamma_\delta((-\infty, 0]))] \\
&\leq \exp(-\lambda\, \delta^{\frac{1}{2}-\eta} + \frac{\lambda^2}{2} \delta \langle \Gamma_0, 1 \rangle)\, \exp\Big(\lambda\big(\Gamma_0((-\infty, 0]) - \int dz g(z) P_z[B_\delta \leq 0]\big)\Big).
\end{aligned}$$

Note that for every $\varepsilon > 0$,

$$\Big| \int_{-\infty}^0 dz g(z) - \int dz g(z) P_z[B_\delta \leq 0] \Big| \leq C_\varepsilon \delta^{\frac{1}{2}-\varepsilon},$$

with a constant C_ε depending only on ε and A. By choosing $\lambda = \gamma^{-\frac{1}{2}+\varepsilon}$ with $0 < \varepsilon < \eta$, we arrive at the desired estimate for $P[\Gamma_\delta((-\infty, 0]) \leq \Gamma_0((-\infty, 0]) - \delta^{\frac{1}{2}-\eta}]$. Slightly different

arguments apply to $P[\Gamma_\delta((-\infty,0]) \geq \Gamma_0((-\infty,0]) + \delta^{\frac{1}{2}-\eta}]$. In fact, it is easier to observe that

$$\begin{aligned} &P[\Gamma_\delta((-\infty,0]) \geq \Gamma_0((-\infty,0]) + \delta^{\frac{1}{2}-\eta}] \\ &\leq P[\langle \Gamma_\delta, 1\rangle \geq \langle \Gamma_0, 1\rangle + \frac{1}{2}\delta^{\frac{1}{2}-\eta}] + P[\Gamma_\delta((0,\infty)) \leq \Gamma_0((0,\infty)) - \frac{1}{2}\delta^{\frac{1}{2}-\eta}]. \end{aligned} \tag{A3}$$

We have just shown how to bound the second term on the right hand side of (A3). As for the first term, we need simply recall that $\langle \Gamma_t, 1\rangle$ is a Feller diffusion and use the fact that for $\lambda \in (0, \frac{2}{\delta})$

$$E[\exp(\lambda\langle \Gamma_\delta, 1\rangle)] = \exp\Big(\frac{\lambda\langle \Gamma_0, 1\rangle}{1 - \frac{1}{2}\lambda\delta}\Big). \tag{A4}$$

This immediately leads to the estimate needed to complete the proof of (A2) and (A1).

From (A1) and the Borel-Cantelli lemma, we get that a.s. there is an integer $n_0(\omega)$ such that, for every $n \geq n_0$, for every $t \geq 0$ of the form $t = j2^{-n}$ and every $z \in \mathbb{R}$ of the form $z = k2^{-n}$, we have

$$|X_{t+2^{-n}}((-\infty, z]) - X_t((-\infty, z])| \leq (2^{-n})^{\frac{1}{2}-\eta}.$$

Note that for every fixed z, the process $t \to X_t((-\infty, z])$ has continuous sample paths a.s. (see e.g. Corollary 6 in [P2]). The proof of Lemma 5.3 is easily completed thanks to this observation, the preceding bound and the usual chaining argument. □

Proof of Lemma 5.8. This is very similar to the proof of (A1) above. Note that the process $(X^*_{t+r}, 0 \leq r \leq \delta)$ is a super-Brownian motion started at X^*_t, which is simply the restriction of X_t to $[z-\gamma, z+\gamma]$. Thanks to this observation and the definition of $E(\delta)$, we see that it is enough to prove the following statement. Let $\Gamma = (\Gamma_r, r \geq 0)$ be super-Brownian motion with initial value $\Gamma_0(dz) = g(z)dz$. Assume that the function g vanishes outside $[-\gamma,\gamma]$ and that $c \leq g(z) \leq c^{-1}$ and $|g(z) - g(z')| \leq |z-z'|^{\frac{1}{2}-\rho}$ for all $z, z' \in [-\gamma, \gamma]$. Then,

$$P[\Gamma_\delta((-\infty,0]) \geq \Gamma_0((-\infty,0]) + \delta^{\frac{3}{4}-\eta''}] \leq \overline{C}\exp(-\delta^{-\overline{\kappa}}), \tag{A5}$$

where the constants $\overline{C}$ and $\overline{\kappa}$ depend only on c, η', η'' and ρ.

In a way similar to (A3) we first write

$$\begin{aligned} &P[\Gamma_\delta((-\infty,0]) \geq \Gamma_0((-\infty,0]) + \delta^{\frac{3}{4}-\eta''}] \\ &\leq P[\langle \Gamma_\delta, 1\rangle \geq \langle \Gamma_0, 1\rangle + \frac{1}{2}\delta^{\frac{3}{4}-\eta''}] + P[\Gamma_\delta((0,\infty)) \leq \Gamma_0((0,\infty)) - \frac{1}{2}\delta^{\frac{3}{4}-\eta''}]. \end{aligned}$$

Thanks to (A4), we see that, for $\lambda < 2/\delta$,

$$P[\langle\Gamma_\delta, 1\rangle \geq \langle\Gamma_0, 1\rangle + \frac{1}{2}\delta^{\frac{3}{4}-\eta''}] \leq \exp(-\lambda(\langle\Gamma_0, 1\rangle + \frac{1}{2}\delta^{\frac{3}{4}-\eta''}))E[e^{\lambda\langle\Gamma_\delta,1\rangle}]$$
$$= \exp(-\frac{\lambda}{2}\delta^{\frac{3}{4}-\eta''}) \exp\Big(\frac{\lambda^2\langle\Gamma_0, 1\rangle\delta/2}{1-\frac{1}{2}\lambda\delta}\Big).$$

Since $\langle\Gamma_0, 1\rangle \leq 2c^{-1}\gamma = 2c^{-1}\delta^{\frac{1}{2}-\eta'}$, we get a bound of the desired form by taking $\lambda = \delta^{-\frac{3}{4}+\varepsilon}$ with $\eta'' > \varepsilon > \eta'$.

For the other term, we proceed as in the proof of Lemma 5.3:

$$P[\Gamma_\delta((0,\infty)) \leq \Gamma_0((0,\infty)) - \frac{1}{2}\delta^{\frac{3}{4}-\eta''}] \leq \exp(\lambda(\Gamma_0((0,\infty)) - \frac{1}{2}\delta^{\frac{3}{4}-\eta''}))E[e^{-\lambda\Gamma_\delta((0,\infty))}],$$

and $E[e^{-\lambda\Gamma_\delta((0,\infty))}] = \exp(-\langle\Gamma_0, u_\delta\rangle)$, with $u_\delta(y) \geq \lambda P_y[B_\delta > 0] - \frac{1}{2}\lambda^2\delta$. It follows that

$$\begin{aligned}
&P[\Gamma_\delta((0,\infty)) \leq \Gamma_0((0,\infty)) - \frac{1}{2}\delta^{\frac{3}{4}-\eta''}] \\
&\leq \exp(-\frac{1}{2}\lambda\delta^{\frac{3}{4}-\eta''} + \frac{\lambda^2}{2}\delta\langle\Gamma_0, 1\rangle) \exp\Big(\lambda\big(\int_0^\infty dzg(z) - \int dzg(z)P_z[B_\delta > 0]\big)\Big) \\
&\leq \exp(-\frac{1}{2}\lambda\delta^{\frac{3}{4}-\eta''} + c^{-1}\lambda^2\delta\gamma) \exp(4\lambda\gamma^{\frac{3}{2}-\rho}),
\end{aligned}$$

where in the last line we used our assumption that $|g(z) - g(0)| \leq |z|^{\frac{1}{2}-\rho}$ to bound $\int_0^\infty dzg(z) - \int dzg(z)P_z[B_\delta > 0]$. In view of the assumptions of Lemma 5.8, we can now choose $\lambda = \delta^{\frac{3}{4}-\varepsilon}$, with $\eta'' > \varepsilon > \frac{3}{2}\eta' + \frac{\rho}{2}$, and we arrive at a bound of the desired form. This completes the proof. $\square$

REFERENCES

[AT] R. Adler and R. Tribe, Uniqueness for a historical SDE with a singular interaction. J. Theoret. Probab. 11, 515–533 (1998)

[BHM] K. Burdzy, R. Hołyst and P. March, A Fleming-Viot particle representation of Dirichlet Laplacian (preprint)

[Da] D.A. Dawson, Measure-valued Markov processes. Ecole d'Eté de Probabilités de Saint-Flour XXI—1991, 1–260, Lecture Notes in Math., 1541, Springer, Berlin, 1993.

[DF] D.A. Dawson, and K. Fleischmann, A continuous super-Brownian motion in a super-Brownian medium. J. Theoret. Probab. 10, 213–276 (1997)

[DIP] D.A. Dawson, I. Iscoe, and E.A. Perkins, Super-Brownian motion: path properties and hitting probabilities. Probab. Th. Rel. Fields 83, 135–205 (1989)

[DP] D.A. Dawson, and E.A. Perkins, Historical processes. Mem. Amer. Math. Soc. 93, no. 454 (1991)

[De] J.F. Delmas, Super-mouvement brownien avec catalyse. Stochastics Stochastics Rep. 58, 303–347 (1996)

[DGL1] D. Dürr, S. Goldstein, and J.L. Lebowitz, Asymptotics of particle trajectories in infinite one-dimensional systems with collisions. Comm. Pure Appl. Math. 38, 573–597 (1985)

[DGL2] D. Dürr, S. Goldstein, and J.L. Lebowitz, Self-diffusion in a nonuniform one-dimensional system of point particles with collisions. Probab. Th. Relat. Fields 75, 279–290 (1987)

[DE] A. Dvoretzky, P. Erdös, Some problems on random walk in space. Proc. Second Berkeley Symp. on Math. Stat. and Probab., pp. 353-367. University of California Press, Berkeley 1951

[Dy] E.B. Dynkin, An Introduction to Branching Measure-Valued Processes. CRM Monograph Series Vol. 6. American Mathematical Society, Providence 1994

[EK] S.N. Ethier and T.G. Kurtz, Markov processes: Characterization and convergence, Wiley, New York, 1986.

[EP] S.N. Evans and E.A. Perkins, Measure-valued branching diffusions with singular interactions. Canadian J. Math. 46, 120–168 (1994)

[G] R. Gisselquist, A continuum of collision process limit theorems. Ann. Probab. 1, 231–239 (1973)

[H] T.E. Harris, Diffusion with "collisions" between particles. J. Appl. Probab. 2, 323–338 (1965).

[Ho] R. Holley, The motion of a large particle. Trans. Amer. Math. Soc. 144, 523–534 (1969)

[KS] N. Konno and T. Shiga, Stochastic partial differential equations for some measure-valued diffusions, Probab. Th. Rel. Fields 79, 201–225 (1988)

[L1] J.F. Le Gall, Marches aléatoires, mouvement brownien et processus de branchement. Séminaire de probabilités XXIII. Lecture Notes Math. 1372, pp. 258–274. Springer 1989.

[L2] J.F. Le Gall, Spatial Branching Processes, Random Snakes and Partial Differential Equations, Lectures in Mathematics ETH Zürich, Bikhäuser 1999

[NP] J. Neveu, J. Pitman, The branching process in a Brownian excursion. Séminaire de probabilités XXIII. Lecture Notes Math. 1372, pp. 248–257. Springer 1989.

[P1] E. Perkins, Polar sets and multiple points for super-Brownian motion. Ann. Probab.

18, 453–491 (1990)

[P2] E. Perkins, On the continuity of measure-valued processes. In: Seminar on Stochastic Processes 1990, pp. 261-268. Progess in Probability 24. Birkhäuser, Boston 1991

[P3] E. Perkins, On the martingale problem for interactive measure-valued diffusions. Memoirs Amer. Math. Soc. 115, no. 549 (1995)

[P4] E. Perkins, Dawson-Watanabe superprocesses and measure-valued diffusions. Ecole d'Eté de Probabilités de Saint-Flour XXIX—1999, Lecture Notes in Math., Springer, to appear

[R] M. Reimers, One dimensional stochastic partial differential equations and the branching measure diffusion. Probab. Th. Rel. Fields 81, 319–340 (1989)

[RY] D. Revuz, M. Yor, Continuous Martingales and Brownian motion. Springer, Berlin 1991

[S] F. Spitzer, Uniform motion with ellastic collision of an infinite particle system. J. Math. Mech. 18, 973–989 (1968/69)

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